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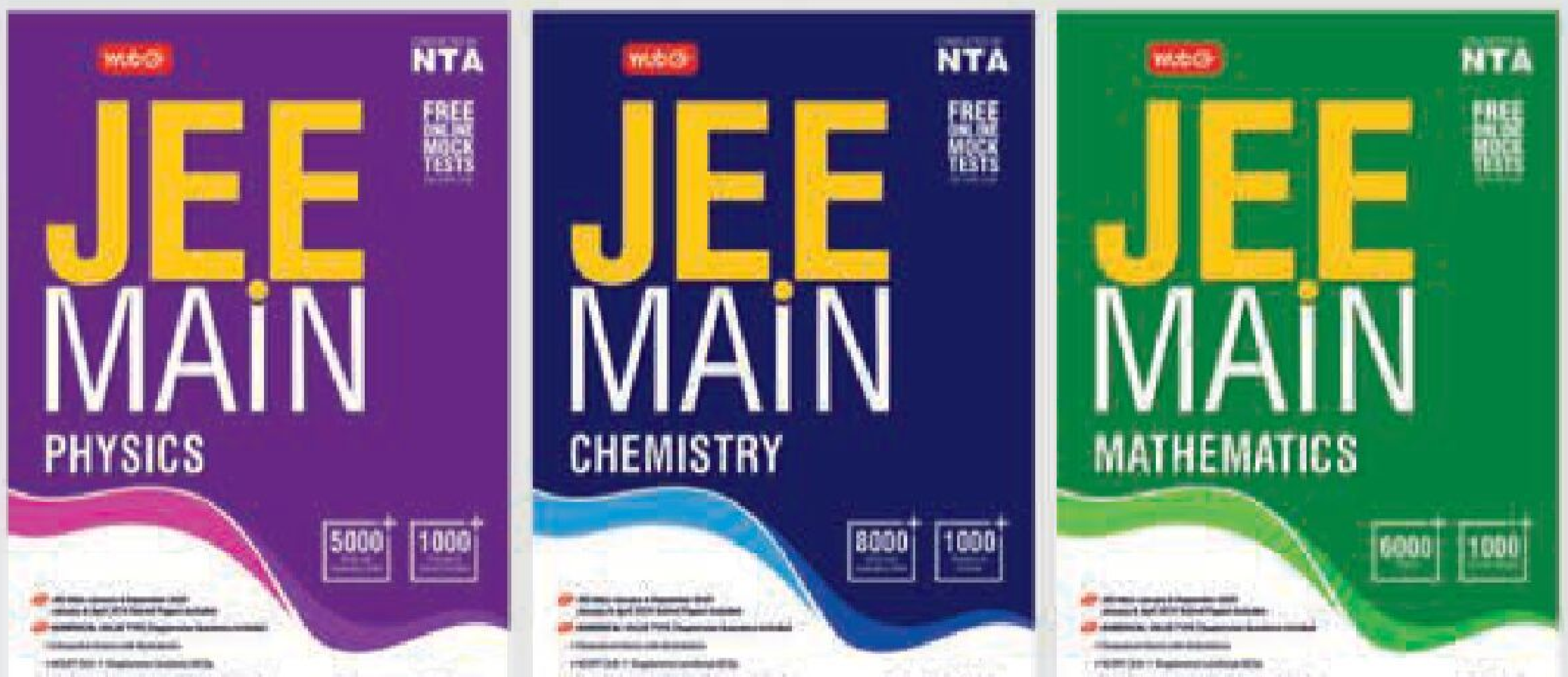
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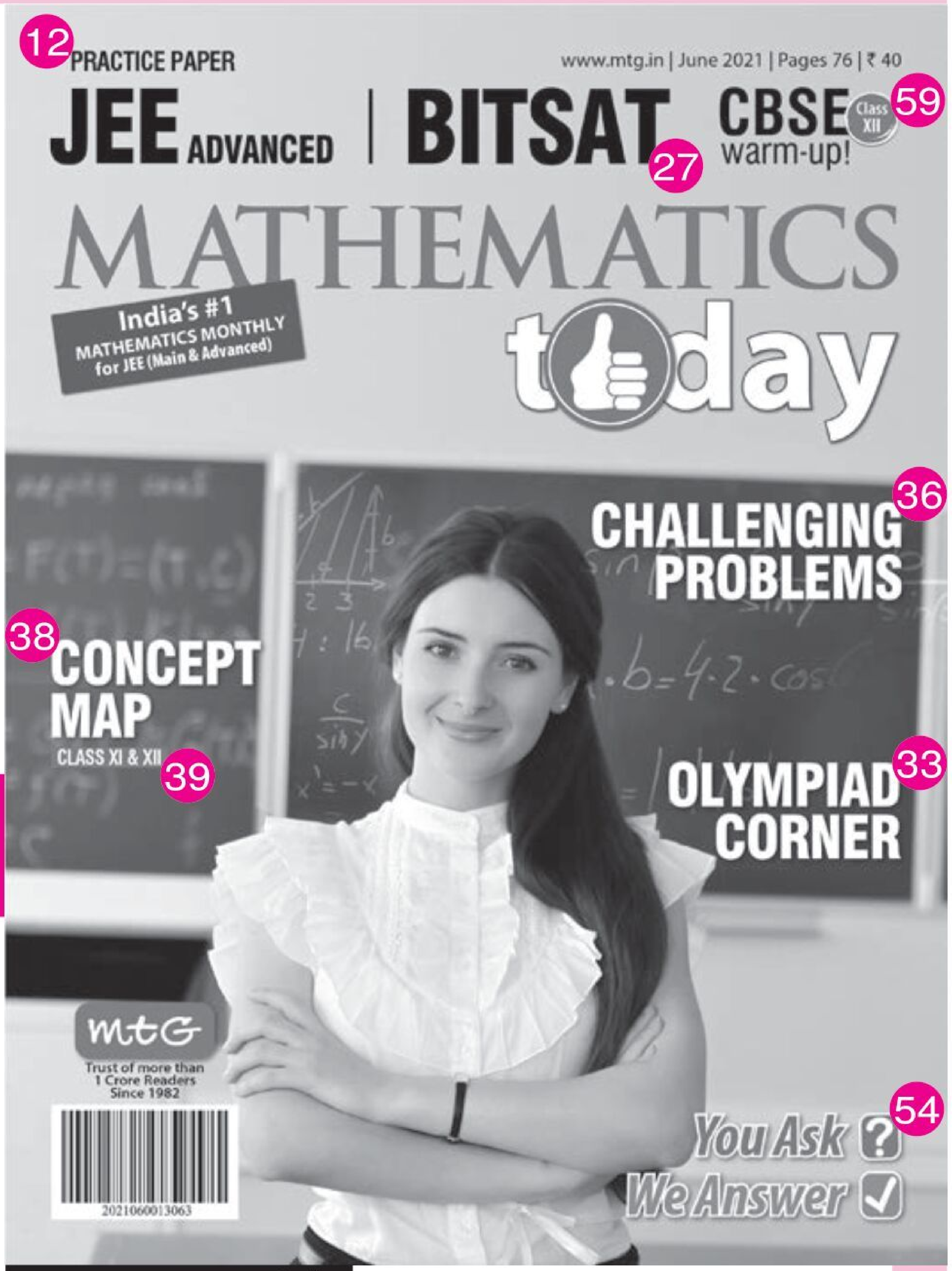
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Exam on 3rd July

JEE
PRACTICE QUESTIONS
ADVANCED 2021

*ALOK KUMAR

ONLY ONE OPTION CORRECT TYPE

1. If $x \in \left(0, \frac{\pi}{2}\right)$, one solution of $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$ is $\frac{\pi}{12}$, the other solution must be $\frac{\lambda\pi}{36}$, then λ is

(a) 11 (b) 12 (c) 13 (d) 14

2. Let two non collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time 't', the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cos t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$ then,

(a) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(b) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(c) $\hat{u}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

(d) $\hat{u}=\frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

3. The roots of $1+z+z^3+z^4=0$ are represented by the vertices of

(a) a square
(b) an equilateral triangle
(c) a rhombus
(d) a rectangle

4. The number of solutions of the equation $\sec^{-1}\left(\frac{4\sqrt{3}x}{x^2+3}\right)+\tan^{-1}\sqrt{6x^2-x^4-9}=\pi/3$ is

(a) 0 (b) 3 (c) 2 (d) 1

5. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors equally inclined to each other at an angle α ($\neq 0$), then the angle between $\vec{a}\times\vec{b}$ and the plane containing $\vec{b}$ and $\vec{c}$ is

(a) $\sin^{-1}\left(\tan\frac{\alpha}{2}.\left|\cot\alpha\right|\right)$

(b) $\cos^{-1}\left(\tan\frac{\alpha}{2}.\left|\cot\alpha\right|\right)$

(c) $\cos^{-1}\left(\cot\frac{\alpha}{2}.\left|\tan\alpha\right|\right)$

(d) $\sin^{-1}\left(\cot\frac{\alpha}{2}.\left|\tan\alpha\right|\right)$

6. The coefficient of x^{3l+2}, in the expansion of $(a+x)^l(b+x)^{l+1}(c+x)^{l+2}$ (l being a positive integer) is

(a) $l(a+b+c)$ (b) $l(a+b+c)+b+2c$
(c) $l(a+b+c)+a+2b+3c$
(d) None of above

7. The solution of the differential equation

$$\frac{dy}{dx}=y+\int_0^1 y\,dx,$$ given that $y=1$, when $x=0$ is

(a) $\frac{2e^x-e+1}{3-e}$ (b) $\frac{2e^x+e+1}{3+e}$

(c) $\frac{2e^x+e+1}{3-e}$ (d) $\frac{e^x-2e+1}{3-e}$

---

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

8. The polynomial function $f(x)$ of degree 6, which satisfies $\lim_{x\to 0}\left(1+\frac{f(x)}{x^3}\right)^{1/x}=e^2$ and has local maximum at $x = 1$ and local minimum at $x = 0$ and $x = 2$ is

(a) $f(x)=2x^4-12x^5+\frac{2}{3}x^6$

(b) $f(x)=2x^4-\frac{12}{5}x^5+2x^6$

(c) $f(x)=2x^4-\frac{12}{5}x^5+\frac{2}{3}x^6$

(d) $f(x)=x^4+\frac{12}{5}x^5-\frac{2}{3}x^6$

9. A normal to the parabola $y^2 = 4ax$ with slope 'm' touches the rectangular hyperbola $x^2 - y^2 = a^2$, if

(a) $m^6 + 4m^4 - 3m^2 + 1 = 0$

(b) $m^6 - 4m^4 + 3m^2 - 1 = 0$

(c) $m^6 + 4m^4 + 3m^2 + 1 = 0$

(d) $m^6 - 4m^4 - 3m^2 + 1 = 0$

10. A periodic function with period '1' is integrable over any finite interval. Also for two real numbers a, b and for two unequal non-zero positive integers m and n, $\int_a^{a+n} f(x)dx=\int_b^{b+m} f(x)dx$,

then the value of $\int_m^n f(x)dx$ is

(a) 0 (b) 1/2 (c) 2 (d) mn

11. If $y=\cos^{-1}\sqrt{\left(\frac{\cos 3x}{\cos^3 x}\right)}$, then $\frac{dy}{dx}$ is

(a) $\sqrt{\frac{3}{\cos 4x+\cos 2x}}$ (b) $\sqrt{\frac{2}{\cos 4x+\cos 2x}}$

(c) $\sqrt{\frac{6}{\cos 4x+\cos 2x}}$ (d) $\sqrt{\frac{6}{\cos 4x-\cos 2x}}$

12. Three numbers are chosen at random from 1, 2, ..., 30. What is the probability that they will form a G.P.?

(a) $\frac{12}{4060}$ (b) $\frac{10}{4060}$ (c) $\frac{19}{4060}$ (d) $\frac{20}{4060}$

13. The set of equations

$\lambda x - y + (\cos\theta)z = 0$

$3x + y + 2z = 0$

$(\cos\theta)x + y + 2z = 0$

$0 \le \theta < 2\pi$, has non–trivial solutions

(a) for no value of λ and θ

(b) for all values of λ and θ

(c) for all values of λ and only two values of θ

(d) for only one value of λ and all values of θ.

14. The area of the smaller region bounded by the curves $x^2 + y^2 = 5$ and $y^2 = 4x$ is

(a) $\frac{\pi}{4}+\frac{1}{6}$ (b) $\frac{\pi}{4}-\frac{1}{6}$

(c) $2\left(\frac{1}{3}-\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)$ (d) $2\left(\frac{1}{3}+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)$

15. The mean daily profit made by a shopkeeper in a month of 30 days was ₹ 350. If the mean profit for the first twenty days was ₹ 400, then the mean profit for the last 10 days would be

(a) ₹ 200 (b) ₹ 250 (c) ₹ 800 (d) ₹ 300

16. If $f(x)$ is a continuous function satisfying $f(x)f(1/x) = f(x) + f(1/x)$ and $f(1) > 0$, then $\lim_{x\to 1} f(x)$ is equal to

(a) 2 (b) 1

(c) 3 (d) none of these

17. Let $f(x)=\begin{cases}(x+1)^3, & -2<x\le -1\\ x^{2/3}-1, & -1<x\le 1\\ -(x-1)^2, & 1<x<2\end{cases}$

The total number of maxima and minima of $f(x)$ is

(a) 4 (b) 3 (c) 2 (d) 1

18. The number of integer values of p for which the vectors $\hat{i}+p^2\hat{j}+p^2\hat{k}$, $p^2\hat{i}+\hat{j}+p^4\hat{k}$ and $p^4\hat{i}+p^4\hat{j}+\hat{k}$ are coplanar is

(a) 8 (b) 4 (c) 2 (d) 0

19. $x^2 + y^2 + 6x + 8y = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are the equations of the two circles. Equation of one of their common tangent is

(a) $7x - 5y - 1 - 5\sqrt{74} = 0$

(b) $7x - 5y - 1 + 5\sqrt{74} = 0$

(c) $7x - 5y + 1 - 5\sqrt{74} = 0$

(d) $5x - 7y + 1 - 5\sqrt{74} = 0$

20. A function $f : R \to R$ satisfies the equation $f(x)f(y) - f(xy) = x + y\ \forall\ x, y \in R$ and $f(1) > 0$, then

(a) $f(x)f^{-1}(x) = x^2 - 4$ (b) $f(x)f^{-1}(x) = x^2 - 6$

(c) $f(x)f^{-1}(x) = x^2 - 1$ (d) none of these

21. Let S be an interval in which $f(x)=\int_0^x \frac{dt}{1+t^3}$ is an invertible function. Suppose g is the inverse of f, then which of the following statement is true?

(a) $g'(x)=\frac{1}{f(f'(x))}$ (b) $g'(x)=\frac{1}{g(g'(x))}$

(c) $g'(x)=\frac{1}{g(f'(x))}$ (d) $g'(x)=\frac{1}{f'(g(x))}$

22. Let f and g are the two functions symmetric about the line $x = a$ and $h(x)$ be a function such that $3h(x) - 4h(2a - x) = 5$. Then the value of $\dfrac{\int_0^{2a} f(x)\cdot g(x)\cdot h(x)\,dx}{\int_0^{2a} f(x)\cdot g(x)\,dx}$ is

(a) -2 (b) 5 (c) 3 (d) -5

23. Let a relation R in the set N of natural numbers be defined as $(x, y) \in R$ if and only if $x^2 - 4xy + 3y^2 = 0$ for all $x, y \in N$. Then the relation R is

(a) Reflexive (b) Symmetric

(c) Transitive (d) Equivalence

24. If $\int_0^{n\pi+t} |\sin x|\,dx = k - \cos t\ (0 \le t \le \pi)$, then the value of k is

(a) $2n$ (b) $4n + 1$

(c) $3n + 1$ (d) $2n + 1$

25. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ and $f(0) = 1$ and $g(x)$ be a function satisfying $f(x)+g(x)=x^2$. If the value of $\int_0^1 f(x)g(x)\,dx - e + \frac{e^2}{2}$ is k, then the value of $|2k|$ is

(a) 1 (b) 2 (c) 3 (d) 4

26. Let $f(x) = [x] - x$ (where $[\cdot]$ is a greatest integer function) and $g(x)=\lim\limits_{n\to\infty}\frac{\{f(x)\}^{4n}-1}{\{f(x)\}^{4n}+1}$ then

$\int \sum_{r=1}^{2014} \{g(x)\}^r\,dx =$

(a) 2014 (b) 2015

(c) 0 (d) None of these

27. If α, β, γ be the angles made by a line with axes so that $2\left(\frac{\tan^2\alpha}{1+\tan^2\alpha}+\frac{\tan^2\beta}{1+\tan^2\beta}+\frac{\tan^2\gamma}{1+\tan^2\gamma}\right)=3\sec^2\frac{\theta}{2}$, then $\theta =$

(a) $\pi/12$ (b) $\pi/10$ (c) $\pi/6$ (d) $\pi/3$

28. If A and B are square matrices of the same order, then which of the following is always true?

(a) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

(b) A and B are non-zero and $|AB| = 0 \Leftrightarrow |A| = 0$ and $|B| = 0$

(c) $(AB)^{-1} = A^{-1}B^{-1}$

(d) $(A + B)^{-1} = A^{-1} + B^{-1}$

29. If A, B and C are three events such that $P(B)=\frac{3}{4}$, $P(A\cap B\cap C')=\frac{1}{3}$ and $P(A'\cap B\cap C')=\frac{1}{3}$, then $P(B \cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

30. The differential equation satisfied by all the circles with centres on the line $x = y$ is

(a) $(x - y)y_2 = (1+ y_1)(1 + y_1^2)$

(b) $(x - y)y_2 = (1 + y_1)(1 + y_1)^2$

(c) $(x - y)y_1^2 = (1 + y_1^2)$

(d) $(x - y)y_1^2 = (1 + y_1)y_2^2$

SOLUTIONS

1. (a) : $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}, \cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore\ \frac{\sin 15^\circ}{\sin x}+\frac{\cos 15^\circ}{\cos x}=2$

$\Rightarrow\ \sin(15^\circ + x) = \sin 2x = \sin(180^\circ - 2x)$

$\Rightarrow x=55^\circ=\frac{11\pi}{36} \Rightarrow \lambda=11$

2. (a) : Given, $\overrightarrow{OP}=\hat{a}\cos t+\hat{b}\sin t$

$\Rightarrow |\overrightarrow{OP}|=\sqrt{(\cos^2 t+\sin^2 t+2\hat{a}\cdot\hat{b}\sin t\cos t)}$

$=\sqrt{1+\hat{a}\cdot\hat{b}\,\sin 2t}$

$\because\ \sin 2t$ is maximum at $t = \pi/4$

$\therefore\ |\overrightarrow{OP}|_{\max} = M = \sqrt{1+\hat{a}\hat{b}}$

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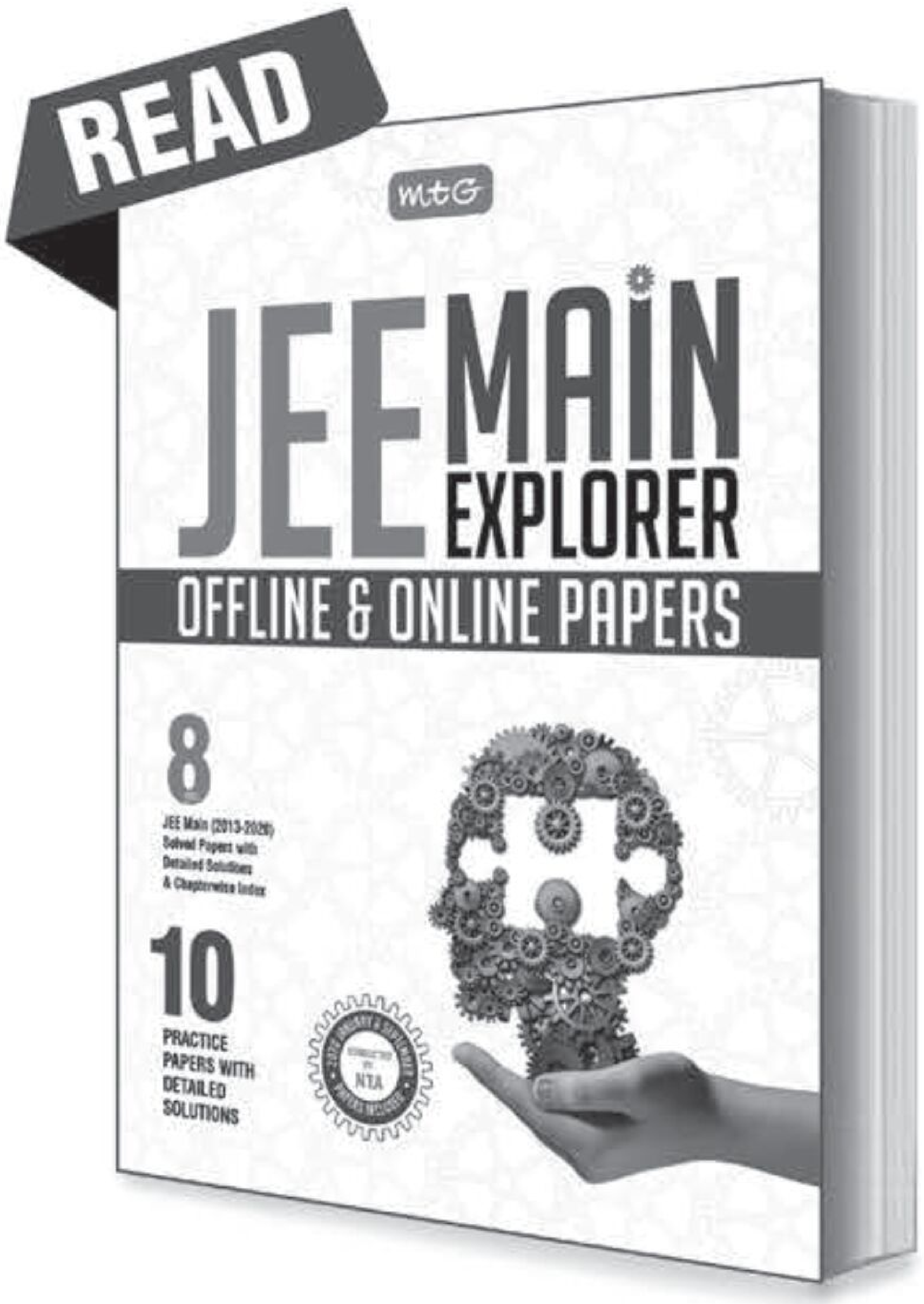
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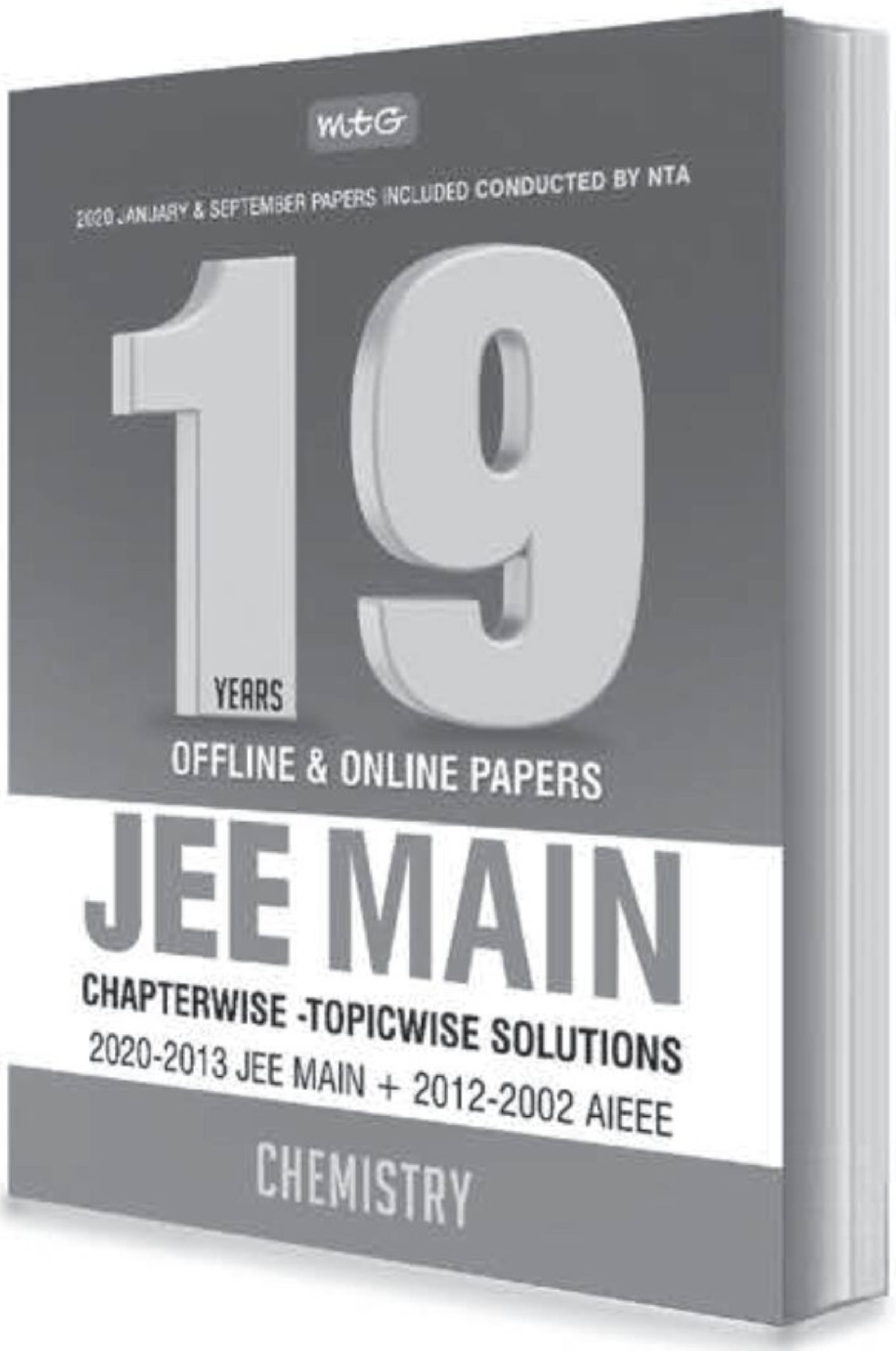
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$$\hat{u}=\frac{\hat{a}\frac{1}{\sqrt{2}}+\hat{b}\frac{1}{\sqrt{2}}}{\left|\hat{a}\frac{1}{\sqrt{2}}+\hat{b}\frac{1}{\sqrt{2}}\right|}=\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

3. (b) : The given equation is $(1 + z)(1 + z^3) = 0$. The distinct roots being $-1, -\omega, -\omega^2$, where ω is a cube root of unity.

$\therefore$ Distinct roots represents by the points

$A(-1, 0)$, $B\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $C\left(\frac{1}{2},\frac{-\sqrt{3}}{2}\right)$

clearly, $AB = BC = CA = \sqrt{3}$

$\therefore$ The three points represent the vertices of an equilateral triangle.

4. (d) : $-9 + 6x^2 - x^4 = -(x^2 - 3)^2$

$\Rightarrow$ $\tan^{-1}$ will be defined only for $x = \pm\sqrt{3}$.

For $x=\sqrt{3}$, it is $\frac{\pi}{3}$.

5. (a) : Let θ be the required angle, then

$$\sin\theta=\cos(90°-\theta)=\left|\frac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})}{|\vec{a}\times\vec{b}||\vec{b}\times\vec{c}|}\right|$$

$$=\left|\frac{[(\vec{a}\times\vec{b})\times\vec{b}]\cdot\vec{c}}{\sin^2\alpha}\right|=\left|\frac{[(\vec{a}\cdot\vec{b})\vec{b}-(\vec{b}\cdot\vec{b})\vec{a}]\cdot\vec{c}}{\sin^2\alpha}\right|$$

$$=\left|\frac{\cos\alpha\,(\vec{b}\cdot\vec{c})-(\vec{a}\cdot\vec{c})}{\sin^2\alpha}\right|=\left|\frac{\cos^2\alpha-\cos\alpha}{\sin^2\alpha}\right|$$

$$=\left|-\frac{\cos\alpha(1-\cos\alpha)}{\sin\alpha\sin\alpha}\right|=\left|-\frac{\cos\alpha\cdot 2\sin^2(\alpha/2)}{\sin\alpha\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right|$$

$$=\left|-\cot\alpha\tan\frac{\alpha}{2}\right|=\left|\cot\alpha\tan\frac{\alpha}{2}\right|$$

$$\therefore\quad \theta=\sin^{-1}\left(\tan\frac{\alpha}{2}|\cot\alpha|\right)$$

6. (b) : Given expression = $(x + a)$ $(x + a)$ . . . l times $\times$ $(x + b)$ $(x + b)$ ....(l + 1) times $\times$ $(x + c)$ $(x + c)$ . . . (l + 2) times

$$=x^{3l+3}+[al+b(l+1)+c(l+2)]\,x^{3l+2}+\ldots$$

$\Rightarrow$ coefficient of x^{3l+2} is $(a + b + c)l + b + 2c$

7. (a) : Let $y=f(x)\Rightarrow\frac{f''(x)}{f'(x)}=1$

$\Rightarrow$ $f(x)=(Ce^{x}+D)$

At $x = 0$, $y=f(x)=1\Rightarrow C=\frac{2}{3-e}$, $D=1-\frac{2}{3-e}$

8. (c): Let $f(x)=ax^4+bx^5+cx^6$

At $a=2$, $f'(x)=x^3(8+5bx+6cx^2)$

Given, $f'(x) = 0$, when $x = 0, 1, 2$

$\Rightarrow$ $f'(1) = 8 + 5b + 6c = 0$

and $f'(2) = 4 + 5b + 12c = 0$

Solving these equations, we get

$b=-12/5$, $c=2/3$

9. (c): Equation of the normal to the parabola $y^2 = 4ax$ with slope m is $y = mx - 2am - am^3$. It touches the rectangular hyperbola $x^2 - y^2 = a^2$ if

$(-2am - am^3)^2 = a^2(m^2 - 1)$

$\Rightarrow$ $a^2m^2(2 + m^2)^2 = a^2(m^2 - 1)$

$\Rightarrow$ $m^2(m^4 + 4m^2 + 4) = m^2 - 1$

$\Rightarrow$ $m^6 + 4m^4 + 3m^2 + 1 = 0$

10. (a) : $f(1 + x) = f(x)$

$$\therefore\quad \int_a^{a+n} f(x)\,dx=\int_0^n f(x)\,dx=n\int_0^1 f(x)\,dx$$

$$\text{similarly}\quad \int_b^{b+m} f(x)dx=m\int_0^1 f(x)\,dx$$

$$\therefore\quad \int_a^{a+n} f(x)dx=\int_b^{b+m} f(x)dx$$

$$\Rightarrow\quad (n-m)\int_0^1 f(x)dx=0\ \Rightarrow\ \int_0^1 f(x)dx=0$$

$$\int_m^n f(x)dx=\int_0^{n-m} f(m+x)\,dx=\int_0^{n-m} f(x)\,dx$$

$$=(n-m)\int_0^1 f(x)\,dx=0$$

11. (c): Given, $y=\cos^{-1}\left(\frac{\sqrt{\cos 3x}}{\sqrt{\cos^3 x}}\right)$

$$\Rightarrow\quad \cos y=\frac{\sqrt{\cos 3x}}{\sqrt{\cos^3 x}}=\frac{\sqrt{4\cos^3 x-3\cos x}}{\cos^3 x}$$

$\Rightarrow$ $\cos^2 y = 1 - 3\tan^2 x$ $\Rightarrow$ $\sin^2 y = 3\tan^2 x$

$\Rightarrow$ $\sin y=\sqrt{3}.\tan x$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{\sqrt{3}\sec^2 x}{\cos y}=\sqrt{\frac{3}{\cos^2 y\cos^4 x}}$$

$$= \sqrt{\frac{3}{(1-3\tan^2 x)\cos^4 x}} = \sqrt{\frac{3}{\cos^2 x(\cos^2 x - 3\sin^2 x)}}$$

$$= \sqrt{\frac{3}{\cos^2 x(4\cos^2 x - 3)}}$$

$$= \sqrt{\frac{3}{\cos x(4\cos^3 x - 3\cos x)}} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

$$= \sqrt{\frac{6}{\cos 4x + \cos 2x}}$$

12. (c): Let A be the event that the numbers of the triplets form a G.P.

Since 3 numbers can be selected from 30 numbers in ${}^{30}C_3$ ways, therefore total number of triplets = ${}^{30}C_3$

Now we count the triplets (arranged in increasing order) whose terms form a G.P. by listing them as follows.

Common ratio	Triplet
2	$\{(i, 2i, 4i), 1 \le i \le 7\}$
3	$\{(i, 3i, 9i), 1 \le i \le 3\}$
4	(1, 4, 16)
5	(1, 5, 25)
3/2	(4, 6, 9), (8, 12, 18), (12, 18, 27)
5/2	(4, 10, 25)
4/3	(9, 12, 16)
5/3	(9, 15, 25)
5/4	(16, 20, 25)

Thus among ${}^{30}C_3$ triplets, there are 19 triplets, whose terms form a G.P.

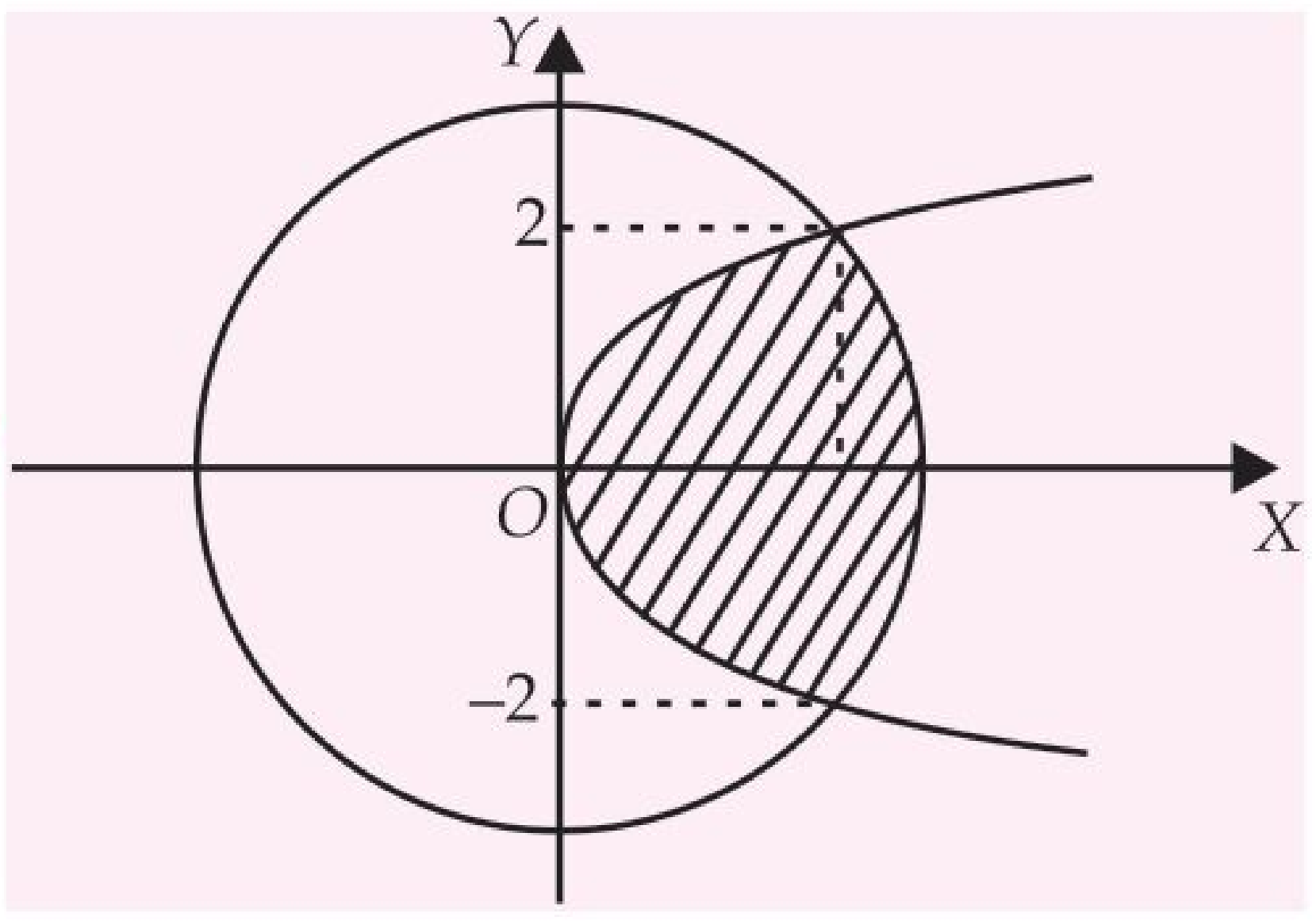
$\therefore$ Required probability, $P(A) = \frac{19}{{}^{30}C_3} = \frac{19}{4060}$

13. (a) **:** Determinant of coefficients

$$= \begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix} = \cos\theta - \cos^2\theta + 6$$

and this is positive for all θ since $|\cos\theta| \le 1$. The only solution is therefore the trivial solution.

14. (d) **:**



$\therefore$ Required area $= \int_{-2}^{2}\left(\sqrt{5-y^2} - \frac{y^2}{4}\right)dy$

$$= 2\int_0^2\left(\sqrt{5-y^2} - \frac{y^2}{4}\right)dy = 2\left(\frac{1}{3} + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

15. (b) **:** $30 \times 350 = 20 \times 400 + 10 \times x$

$\Rightarrow \quad 10x = 10500 - 8000 \Rightarrow 10x = 2500 \Rightarrow x = 250$

16. (a) **:** Since, $f(x)$ is continuous function so,

$\lim_{x\to 1} f(x) = f(1)$.

Putting $x = 1$ in the given equation, we have

$(f(1))^2 = 2f(1)$, so $f(1) = 0$ or 2.

Since $f(1) > 0$, so $f(1) = 2$.

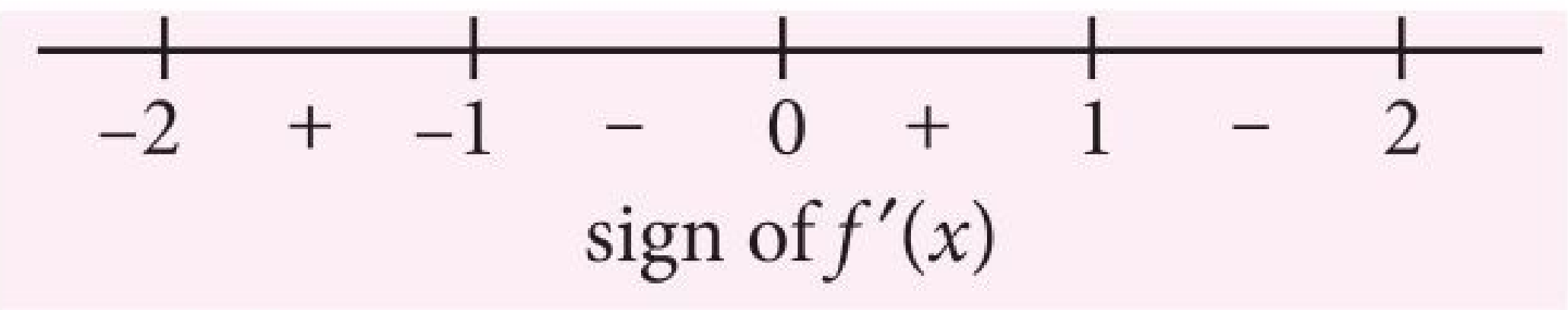
17. (b) **:** $f'(x) = \begin{cases} 3(x+1)^2 & -2 < x < -1 \\ \frac{2}{3} \times (x^{-1/3}) & -1 < x < 1 - \{0\} \\ -2(x-1) & 1 < x < 2 \end{cases}$

SAMURAI SUDOKU

ANSWER · MAY 2021

3	6	9	5	4	2	1	7	8				9	8	1	4	2	7	6	3	5
1	7	4	8	3	6	9	5	2				4	5	7	3	6	8	1	9	2
2	5	8	9	1	7	4	6	3				2	6	3	5	1	9	8	7	4
6	4	2	1	7	5	8	3	9				7	2	9	6	8	3	4	5	1
7	3	5	2	8	9	6	1	4				6	4	8	1	9	5	7	2	3
9	8	1	4	6	3	7	2	5				1	3	5	2	7	4	9	8	6
5	1	6	3	9	8	2	4	7	9	3	5	8	1	6	9	3	2	5	4	7
8	2	7	6	5	4	3	9	1	8	6	4	5	7	2	8	4	1	3	6	9
4	9	3	7	2	1	5	8	6	7	1	2	3	9	4	7	5	6	2	1	8
						8	1	3	5	4	9	2	6	7						
						6	7	2	3	8	1	4	5	9						
						4	5	9	6	2	7	1	8	3						
2	6	1	5	8	9	7	3	4	1	9	8	6	2	5	3	7	9	4	1	8
7	5	4	3	6	1	9	2	8	4	5	6	7	3	1	4	2	8	5	9	6
8	3	9	2	7	4	1	6	5	2	7	3	9	4	8	1	5	6	2	3	7
4	9	8	1	5	6	3	7	2				3	5	6	8	9	1	7	4	2
6	7	2	4	3	8	5	1	9				2	7	9	5	4	3	6	8	1
5	1	3	9	2	7	4	8	6				8	1	4	2	6	7	3	5	9
3	4	7	6	9	2	8	5	1				1	8	2	7	3	5	9	6	4
9	2	5	8	1	3	6	4	7				4	9	3	6	1	2	8	7	5
1	8	6	7	4	5	2	9	3				5	6	7	9	8	4	1	2	3

$f'(x)$ does not exist at $x = -1, 0, 1$



18. (c): Since, given vectors are coplanar,

$$\begin{vmatrix} 1 & p^2 & p^2 \\ p^2 & 1 & p^4 \\ p^4 & p^4 & 1 \end{vmatrix} = 0 \Rightarrow 1 + p^{10} - p^6 - p^4 = 0$$

$\therefore$ $p = 1, -1$ are only two integral solutions.

19. (c): Both the circles have radius = 5 and they intersect each other, therefore their common tangent is parallel to the line joining their centres.

Equation of the line joining their centre is

$7x - 5y + 1 = 0.$

$\therefore$ Equation of the common tangent is $7x - 5y = c$

$$\therefore \left|\frac{c+1}{\sqrt{74}}\right| = 5 \Rightarrow c = \pm 5\sqrt{74} - 1$$

$\therefore$ Equation is $7x - 5y + 1 \mp 5\sqrt{74} = 0.$

20. (c): Taking $x = y = 1$, we get

$f(1)f(1) - f(1) = 2$

$\Rightarrow f^2(1) - f(1) - 2 = 0$

$\Rightarrow (f(1) - 2)(f(1) + 1) = 0$

$\Rightarrow f(1) = 2$ (as $f(1) > 0$)

Taking $y = 1$, we get

$f(x) \cdot f(1) - f(x) = x + 1$

$\Rightarrow f(x) = x + 1 \Rightarrow f^{-1}(x) = x - 1$

$\therefore f(x) \cdot f^{-1}(x) = x^2 - 1$

21. (d) : $\because \; f'(x) = \dfrac{1}{1+x^3}$

Also, $f(g(x)) = \displaystyle\int_0^{g(x)} \frac{dt}{1+t^3}$ $(\because \; g = f^{-1})$

$$\Rightarrow x = \int_0^{g(x)} \frac{dt}{1+t^3}$$

Now, $1 = \dfrac{1}{1+(g(x))^3} g'(x)$

$\Rightarrow g'(x) = 1 + (g(x))^3$

$$= \frac{1}{f'(g(x))}$$

22. (d) : $\because f(a - x) = f(a + x)$

or, $f(x + 2a) = f(-x) \Rightarrow f(x) = f(2a - x)$

Similarly, $g(2a - x) = g(x)$

Now, let $I = \displaystyle\int_0^{2a} f(x) \cdot g(x) \cdot h(x) dx$

$$= \int_0^{2a} f(x) \cdot g(x) \cdot h(2a - x) dx$$

or, $4I = \displaystyle\int_0^{2a} f(x) \cdot g(x) \cdot (3h(x) - 5) dx$

$$= 3I - 5\int_0^{2a} f(x) \cdot g(x) dx$$

$$\therefore I = -5\int_0^{2a} f(x) \cdot g(x) dx$$

$$\Rightarrow \frac{\int_0^{2a} f(x) \cdot g(x) \cdot h(x) dx}{\int_0^{2a} f(x) \cdot g(x) dx} = -5$$

23. (a)

24. (d) : $\displaystyle\int_0^{n\pi+t} |\sin x| dx = \int_0^{n\pi} |\sin x| dx + \int_{n\pi}^{n\pi+t} |\sin x| dx$

$= 2n + (-\cos x)_{n\pi}^{n\pi+t}$

$= 2n + \cos(n\pi) - \cos(n\pi + t)$

$$= \begin{cases} 2n + 1 - \cos t; & n = \text{even} \\ 2n - 1 + \cos t; & n = \text{odd} \end{cases}$$

25. (c): $\because f'(x) = f(x)$ and $f(0) = 1$

$\Rightarrow f(x) = e^x$

$\therefore g(x) = x^2 - e^x$

Now, $\displaystyle\int_0^1 f(x)g(x)\,dx = \int_0^1 e^x(x^2 - e^x)dx = e - \frac{e^2}{2} - \frac{3}{2}$

$$\Rightarrow \int_0^1 f(x)g(x)dx - e + \frac{e^2}{2} = \frac{-3}{2}$$

$$\Rightarrow k = -\frac{3}{2} \Rightarrow |2k| = 3$$

26. (d) **27. (d)**

28. (a)

29. (a) : We have,

$P(B \cap C') = P[(A \cup A') \cap (B \cap C')]$

$$= P(A \cap B \cap C') + P(A' \cap B \cap C') = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

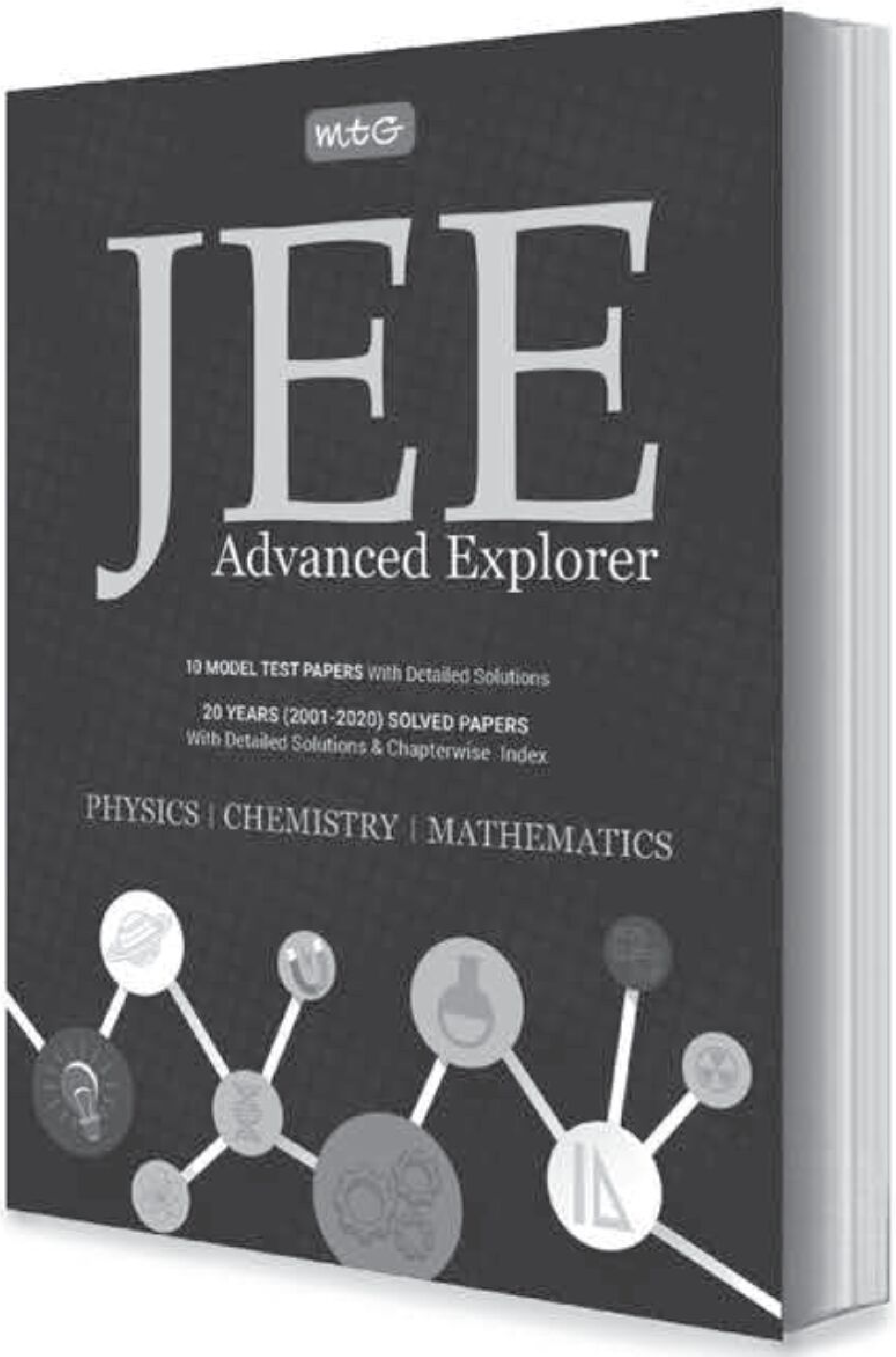
Now, $P(B \cap C) = P(B) - P(B \cap C') = \dfrac{3}{4} - \dfrac{2}{3} = \dfrac{1}{12}$

30. (a)

◆◆

Precision Revision for JEE is Here!

MTG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JEE ADVANCED

PRACTICE PAPER

Exam on
3rd July 2021

2021
PROBLEM
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PROBLEM

PAPER-1

SINGLE OPTION CORRECT TYPE

1. Area of the region bounded by the curves $y = x^2$ and $y = \frac{2}{1+x^2}$ is

(a) $\pi + \frac{2}{3}$ (b) $\pi - \frac{2}{3}$

(c) $\frac{\pi}{2} + \frac{1}{3}$ (d) $2\pi - \frac{2}{3}$

2. Solution of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is

(a) $x = y^2(c + y^2)$ (b) $x = y(c - y^2)$

(c) $x = 2y(c - y^2)$ (d) $x = y(c + y^2)$

3. If z-axis be vertical, then the equation of the line of greatest slope through the point (2, –1, 0) on the plane $2x + 3y - 4z = 1$ is

(a) $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z}{0}$

(b) $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-0}{5}$

(c) $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-0}{12}$

(d) $\frac{x-2}{8} = \frac{y+1}{12} = \frac{z}{13}$

4. Two lines with direction ratios $(\alpha, \sin\alpha, e^{\alpha})$ and $(\cos\pi\alpha, 0, 1)$ are perpendicular to each other. Then α belongs to the interval

(a) $\left(-1, \frac{-1}{2}\right)$ (b) $\left(\frac{-1}{2}, 0\right)$

(c) $(-2, -1)$ (d) $\left(\frac{1}{2}, 1\right)$

5. The number of real roots of the equation $\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3}x) = \frac{-\pi}{2}$ where for any $y \in [-1, 1]$, $\sin^{-1} y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is

(a) 1 (b) 2

(c) 4 (d) infinity

6. A bag contains 3 black and 7 white balls. Three balls are drawn one after other replacing the ball drawn each time along with two more balls of the same colour. If the third drawn ball is black, then the first drawn ball is also black with probability

(a) $\frac{7}{12}$ (b) $\frac{5}{12}$

(c) $\frac{1}{4}$ (d) $\frac{3}{4}$

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. Solution of the differential equation

$$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\sin^2(x^2+y^2)}{y^3}$$ is

(a) $-\cot(x^2+y^2) = \left(\frac{x}{y}\right)^2 + c$

(b) $\frac{y^2}{x^2 + y^2 c} = -\tan^2(x^2+y^2)$

(c) $-\cot(x^2+y^2) = \left(\frac{y}{x}\right)^2 + c$

(d) $\frac{y^2 + x^2 c}{x^2} = -\tan^2(x^2+y^2)$

8. Let P_1 denotes the equation of the plane to which the vector $(\hat{i}+\hat{j})$ is normal and which contains the line L whose equation is $\vec{r}=(\hat{i}+\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}-\hat{k})$, P_2 denotes the equation of the plane containing the line 'L' and a point with position vector $\hat{j}$. Then which of the following is/are not true.

(a) Equation of P_1 is $x + y = 2$.

(b) Equation of P_2 is $\vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})=2$

(c) The acute angle between P_1 and P_2 is 75°

(d) The angle between the plane and the line is 0°

9. The direction cosines of the lines bisecting the angle between the lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 and the angle between these lines is θ are

(a) $\dfrac{l_1+l_2}{\cos\frac{\theta}{2}}, \dfrac{m_1+m_2}{\cos\frac{\theta}{2}}, \dfrac{n_1+n_2}{\cos\frac{\theta}{2}}$

(b) $\dfrac{l_1-l_2}{\sin\frac{\theta}{2}}, \dfrac{m_1-m_2}{\sin\frac{\theta}{2}}, \dfrac{n_1-n_2}{\sin\frac{\theta}{2}}$

(c) $\dfrac{l_1+l_2}{2\cos\frac{\theta}{2}}, \dfrac{m_1+m_2}{2\cos\frac{\theta}{2}}, \dfrac{n_1+n_2}{2\cos\frac{\theta}{2}}$

(d) $\dfrac{l_1-l_2}{2\sin\frac{\theta}{2}}, \dfrac{m_1-m_2}{2\sin\frac{\theta}{2}}, \dfrac{n_1-n_2}{2\sin\frac{\theta}{2}}$

10. Which of the following is/are true?

(a) If y is a function of x such that $y(x-y)^2 = x$ then $\displaystyle\int\frac{dx}{x-3y}=\ln|x-3y|+c$

(b) $\displaystyle\int_0^{2\pi}\cos^3 x\,dx=\frac{8}{3}$

(c) $\displaystyle\int_2^3\left(\sqrt{2x-\sqrt{5(4x-5)}}+\sqrt{2x+\sqrt{5(4x-5)}}\right)dx$

$$=\frac{\sqrt{10}}{2}+\frac{7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}}$$

(d) The length of perpendicular from the point $(1, -1, 2)$ on the line $\dfrac{x+1}{2}=\dfrac{y-2}{-3}=\dfrac{z+2}{4}$ is zero.

11. Let $\vec{a}$, $\vec{b}$ be two unit vectors. If $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$ then $|\vec{v}|=$

(a) $|\vec{u}|$

(b) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$

(c) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$

(d) $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

12. If x, y, z are three real numbers such that $x+y+z=4$ and $x^2+y^2+z^2=6$. Then

(a) $\frac{1}{2}\le x,y,z\le 1$

(b) $\frac{2}{3}\le x,y,z\le 2$

(c) x can take both of the values $\frac{2}{3}$ and 2

(d) cannot take any of the values $\frac{2}{3}$ and 2

NUMERICAL VALUE TYPE

13. Let $f(x) = (x-1)(x-2)(x-3)\ldots(x-n)$, $n\in N$ and $\displaystyle\int\frac{f(x)f''(x)-(f'(x))^2}{f^2(x)}dx=u(x)+c$, where '$c$' is arbitrary constant. If $u(x) = 5$ has 8 solutions, then the number of possible values of 'n' is equal to______.

14. Let $f(x)=\min\{x+1, \sqrt{1-x}\}$ for all $x\le 1$. Then the area bounded by $y=f(x)$ and x-axis is______.

15. Let the function f satisfies $f(-x)f'(x)=f(x)f'(-x)$ for all x and $f(0) = 3$ then $\displaystyle\int_{-2010}^{2010}\frac{dx}{3+f(x)}=$______.

16. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$, $\vec{c}=\hat{i}-\hat{j}$ and $\vec{d}=6\hat{i}+2\hat{j}+3\hat{k}$. If $\vec{d}=x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$ where x, y and z are scalars, then the value of $x+y+z$ is______.

17. The greatest value of $f(x) = |x^2-5x+6|$ on the interval $[0, 2.4]$ is______.

18. Let $a_1 = 1$, $a_2 = 1 + a_1$, $a_3 = 1 + a_1a_2 \ldots$, $a_{n+1} = 1 + a_1a_2 \ldots a_n$, then $\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}+\ldots$ upto ∞ is equal to______.

MONTHLY TEST DRIVE CLASS XII ANSWER KEY

1.	(b)	**2.**	(a)	**3.**	(a)	**4.**	(c)	**5.**	(b)
6.	(c)	**7.**	(a,c)	**8.**	(a)	**9.**	(a)	**10.**	(a,d)
11.	(a,b,c)	**12.**	(a,c)	**13.**	(a,b,d)	**14.**	(a)	**15.**	(c)
16.	(d)	**17.**	(0)	**18.**	(9)	**19.**	(3)	**20.**	(1)

PAPER-2

SINGLE DIGIT INTEGER ANSWER TYPE

1. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is______.

2. f is continuous and differentiable on $[0, 1]$ and $f(0) = 0$ and $f(1) = 1$, then the minimum value of $\int_0^1 (f'(x))^2\,dx$ is______.

3. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of the water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = 1/15$ then time to drain the tank, if the water is 4 m deep to start with, is 12 T minutes where T is equal to______.

4. The number of real ordered pairs (x, y) satisfying the equation $3x^2 - 4xy + 3y^2 + 10x - 10y + 10 = 0$ is______.

5. If the area enclosed by $y^2 = 4ax$ and line $y = ax$ is $\frac{1}{3}$ sq.units, then the positive root of the equation $x^2 + 2x = a$ is______.

6. $$\int \frac{1}{(x^3-8)^2}dx = A\left(\frac{x}{x^3-8}\right) + B\ \log|x-2| + C\ \log|x^2+2x+4| + D\ \tan^{-1}\left(\frac{x+1}{E}\right) + \lambda$$

(where λ is an arbitrary constant), then the value of $A + 2(D)(E)$ is ______.

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. If a curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area bounded by the curve, line $x = 4$ and x-axis is 8 square units, then

(a) $a = 3$ (b) $a = -3$ (c) $b = 1$ (d) $b = -1$

8. The solution of the differential equation $(y^2\,dx - 2xy\,dy) = x^3y^3\,dy + x^2y^4\,dx$ is

(a) $\log\left(\frac{x}{y^2}\right) = \frac{(xy)^2}{2} + c$ (b) $\frac{x}{cy^2} = e^{\frac{(xy)^2}{2}}$

(c) $\left(\frac{x}{y^2}\right) = 2\log(xy) + c$

(d) $\frac{x}{y^2} = e^{2(xy)} + c$

9. Let $\vec{x}, \vec{y}, \vec{z}$ are unit vectors such that $\vec{x} + \vec{y} + \vec{z} = \vec{a}$, $\vec{x} \times (\vec{y} \times \vec{z}) = \vec{b}$, $(\vec{x} \times \vec{y}) \times \vec{z} = \vec{c}$, $\vec{a} \cdot \vec{x} = \frac{3}{2}$, $\vec{a} \cdot \vec{y} = \frac{7}{4}$ and $|\vec{a}| = 2$, then

(a) $\vec{x} = \frac{3\vec{a} + 4\vec{b} + 8\vec{c}}{3}$ (b) $\vec{y} = 4\vec{c}$

(c) $\vec{z} = (\vec{b} - \vec{c})$ (d) $\vec{y} = -4\vec{c}$

10. Consider the circle $C_1 : x^2 + y^2 = 4$ and the circle $C_2 : x^2 + y^2 - 6x - 8y - 24 = 0$. Then

(a) C_1 and C_2 touch each other internally.

(b) C_1 completely lies within C_2 without a common point

(c) C_1 and C_2 touch externally

(d) The number of common tangents is one.

11. If $I_n = \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+e^x)\sin x}dx$ for $n = 0, 1, 2, \ldots$ then

(a) $I_n = I_{n+1}$ (b) $I_n = I_{n+2}$

(c) $\sum_{k=1}^{20} I_k = 10\pi$ (d) $\sum_{k=1}^{10} I_{2k+1} = 10\pi$

12. a, b, c are non zero and distinct numbers such that the sum of any two is not zero. If the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal roots, then

(a) a, b, c are in G.P.

(b) a, b, c are in H.P.

(c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

(d) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in A.P

COMPREHENSION TYPE

Paragraph for Question No. 13 to 15

Consider a square with vertices at $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$. Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge.

13. The region S is surmounted by
(a) a circle (b) a square
(c) four parabolas (d) two ellipses

14. The point where the function, defined by the boundary (in the first quadrant) of the region S is not differentiable, is
(a) $\left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ (b) $(\sqrt{2}-1, \sqrt{2}-1)$
(c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(1-\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right)$

15. The area of the region S is
(a) $\frac{1}{4}(15\sqrt{2}-19)$ sq.units
(b) $\frac{1}{3}(16\sqrt{2}-20)$ sq.units
(c) $\frac{1}{4}(16\sqrt{2}+19)$ sq.units
(d) $16\sqrt{3}+23$ sq.units

Paragraph for Question Nos. 16 to 18

Let f be a function defined in a neighbourhood of a. Then
(i) f is continuous at $x = a$ if and only if

$$\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = f(a)$$

(ii) The left derivative at a denoted by $Lf'(a)$ is defined to be the $\lim_{h\to 0}\left(\frac{f(a-h)-f(a)}{-h}\right)$ and right derivative

$$Rf'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

f is differentiable at $x = a$ if and only if $Lf'(a) = Rf'(a)$.

16. Let $[a]$ denotes the greatest integer less than or equal to a where a is any real. Consider the function $f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2}$. Then
(a) f is discontinuous at infinitely many points
(b) f is continuous for all real x, but the derivatives $f'(x)$ does not exist at many points
(c) f is continuous for all x but f is not differentiable at any point
(d) exists for all

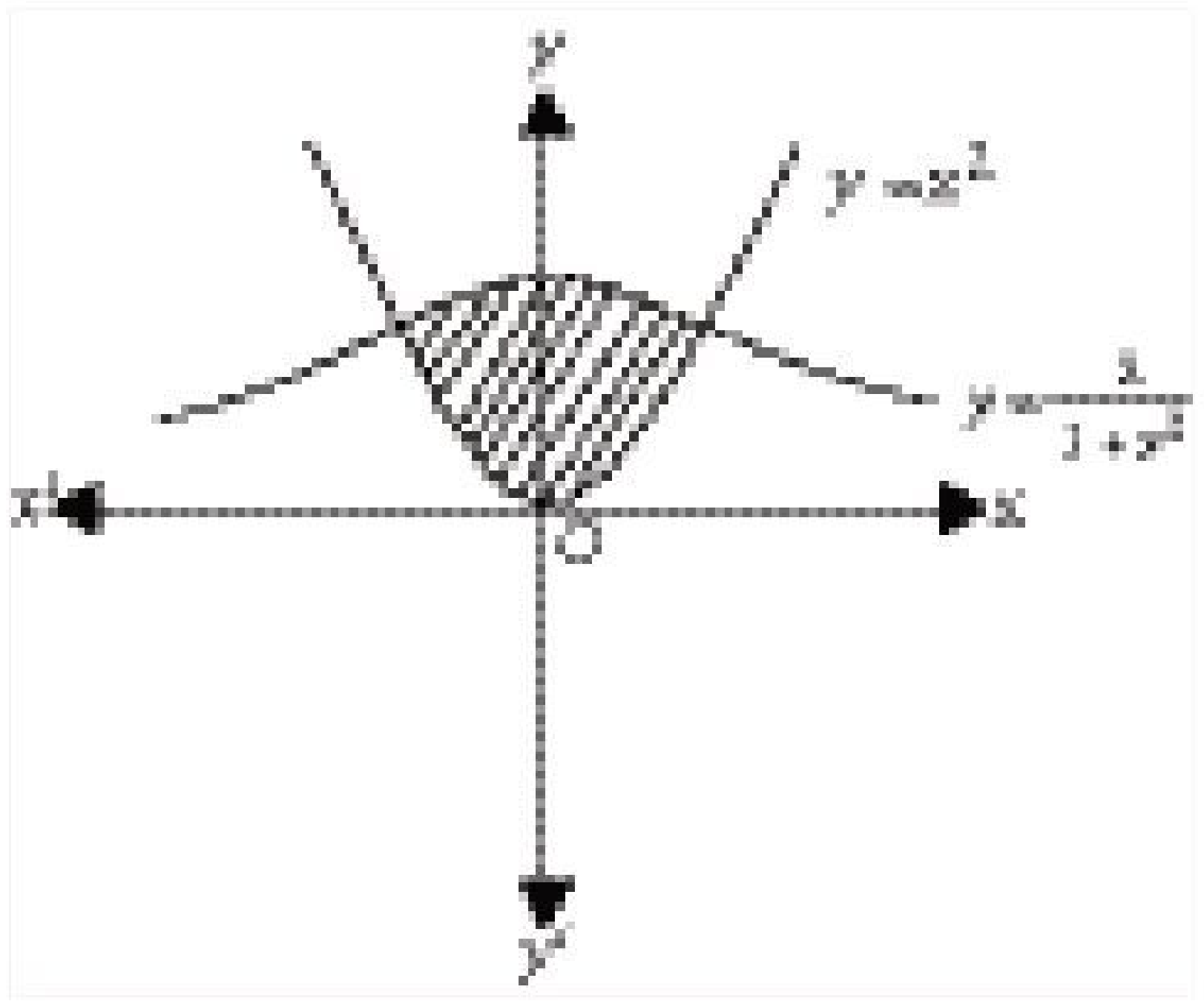
17. Let $f(x) = \begin{cases} \frac{1}{3}-x, & x \le \frac{1}{3} \\ \left(\frac{1}{3}-x\right)^2, & x > \frac{1}{3} \end{cases}$. Then on the interval $[0, 1]$, Lagrange's Mean Value theorem is not applicable to $f(x)$ because
(a) f is not continuous (b) f is not differentiable
(c) $f(0) = f(1)$ (d) $f'(0)$ does not exist

18. $f: R \to R$ satisfying the equation $f(x+y) = f(x)f(y)$ for all real x and y and $f(x) \ne 0$ for all x. If f is differentiable at $x = 0$ and $f'(0) = 2$, then $\log|f(2)|$ is
(a) 1 (b) 2
(c) 4 (d) not defined

SOLUTIONS

PAPER-1

1. (b) : Solving the curves, we get $x = \pm 1$



$$\therefore \text{ Required area} = 2\int_0^1\left(\frac{2}{1+x^2}-x^2\right)dx$$

$$= 2\left[\left[2\tan^{-1}x\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right] = 2\left[2\frac{\pi}{4}-\frac{1}{3}\right] = \left(\pi-\frac{2}{3}\right)\text{sq. units}$$

2. (d) : We have $(x+2y^3)\frac{dy}{dx} = y \Rightarrow y\frac{dx}{dy} = x+2y^3$

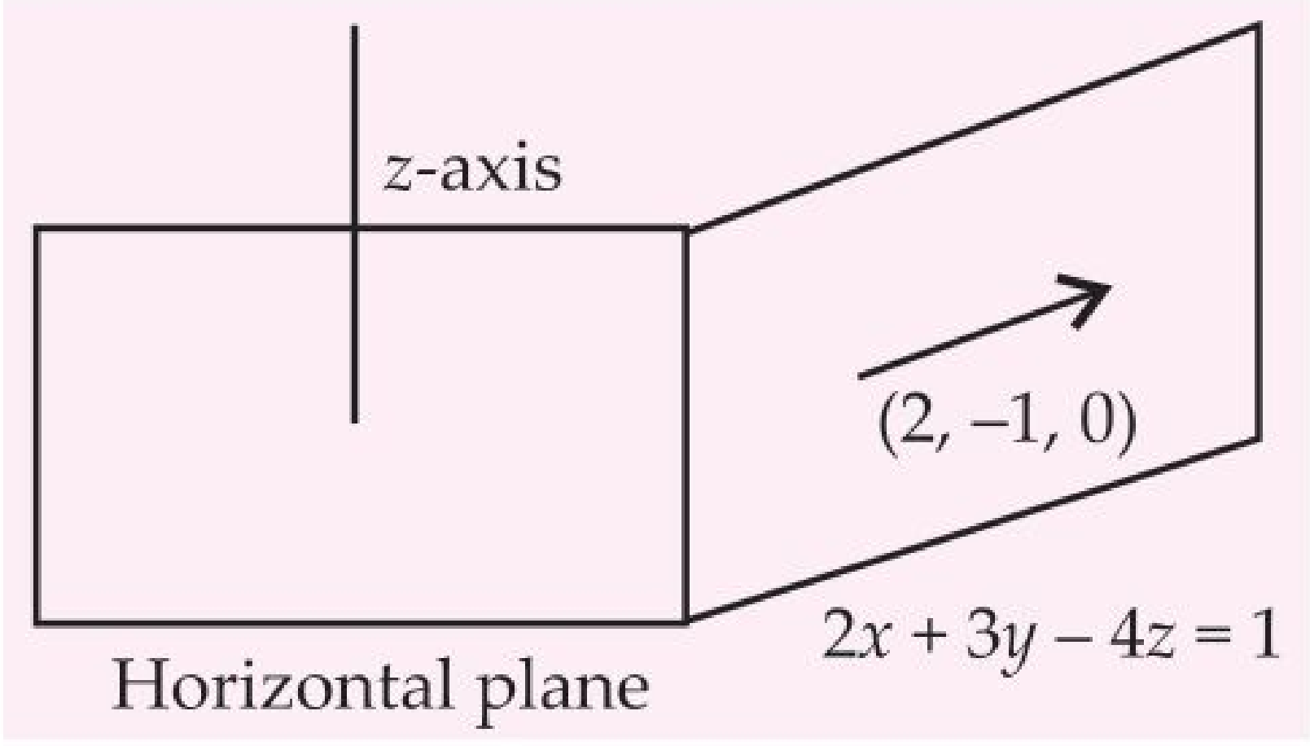
$$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2,$$

which is a linear differential equation. If we take x as the depending variable, then

$$\text{I.F.} = e^{\int P dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$\therefore$ The solution is $x\cdot\frac{1}{y} = y^2 + c \Rightarrow x = y(c+y^2)$

3. (d) : Let equation of the line with greatest slope is $\frac{x-2}{a} = \frac{y+1}{b} = \frac{z}{c}$,
where $2a + 3b - 4c = 0$ ... (i)



We know that equation of horizontal plane is $z = 0$
i.e. $0\cdot x + 0\cdot y + 1\cdot z = 0$

Now, a vector along the line of intersection of given plane and horizontal plane is $\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 2 & 3 & -4 \end{vmatrix} = -3\hat{i} + 2\hat{j}$

Since the line of greatest slope is also perpendicular to the vector $\vec{V}$.

Hence $-3a + 2b = 0$ ... (ii)

From (i) and (ii), $\frac{a}{8} = \frac{b}{12} = \frac{c}{13}$

Equation of the line of greatest slope

$$= \frac{x-2}{8} = \frac{y+1}{12} = \frac{z}{13}$$

4. (c) : $\alpha\cos\pi\alpha + e^{\alpha} = 0$. Let $f(\alpha) = e^{\alpha} + \alpha\cos\pi\alpha$

f is continuous and $f(-2) = e^{-2} - 2 < 0$ and $f(-1) = e^{-1} + 1 > 0$. $\therefore$ $\alpha \in (-2, -1)$

5. (a) : $\sin(\sin^{-1}6x) = \sin\left(-\sin^{-1}(6\sqrt{3}x) - \frac{\pi}{2}\right)$

$$\Rightarrow \quad 6x = -\sqrt{1-108x^2} \Rightarrow \quad 144x^2 = 1 \quad \therefore \quad x = \pm\frac{1}{12}$$

$x = \frac{-1}{12}$ satisfies the initial equation.

6. (b) : Let B_j denotes j^{th} drawn ball is black ($j = 1, 2, 3$) and W_j denotes the j^{th} drawn ball is white ($j = 1, 2, 3$)

Now $B_3 = (B_1 \cap B_2 \cap B_3) \cup (B_1 \cap W_2 \cap B_3) \cup (W_1 \cap B_2 \cap B_3) \cup (W_1 \cap W_2 \cap B_3)$

$$\therefore \ P(B_3) = \frac{3}{10}\times\frac{5}{12}\times\frac{7}{14} + \frac{3}{10}\times\frac{7}{12}\times\frac{5}{14} + \frac{7}{10}\times\frac{3}{12}\times\frac{5}{14} + \frac{7}{10}\times\frac{9}{12}\times\frac{3}{14} = \frac{504}{1680}$$

$$\text{Now, } P\left(\frac{B_1}{B_3}\right) = \frac{(P(B_1 \cap B_2 \cap B_3) + P(B_1 \cap W_2 \cap B_3))}{P(B_3)}$$

$$= \frac{210}{504} = \frac{5}{12}$$

7. (a, b) : The given differential equation can be written as

$$\frac{xdx + ydy}{ydx - xdy} = \frac{x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \quad \frac{2xdx + 2ydy}{ydx - xdy} = \frac{2x\sin^2(x^2+y^2)}{y^3}$$

$$\Rightarrow \quad \frac{d(x^2+y^2)}{\sin^2(x^2+y^2)} = \frac{2x}{y^3}(ydx - xdy)$$

$$\Rightarrow \quad \operatorname{cosec}^2(x^2+y^2)\, d(x^2+y^2) = 2\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right)$$

On integrating, we get

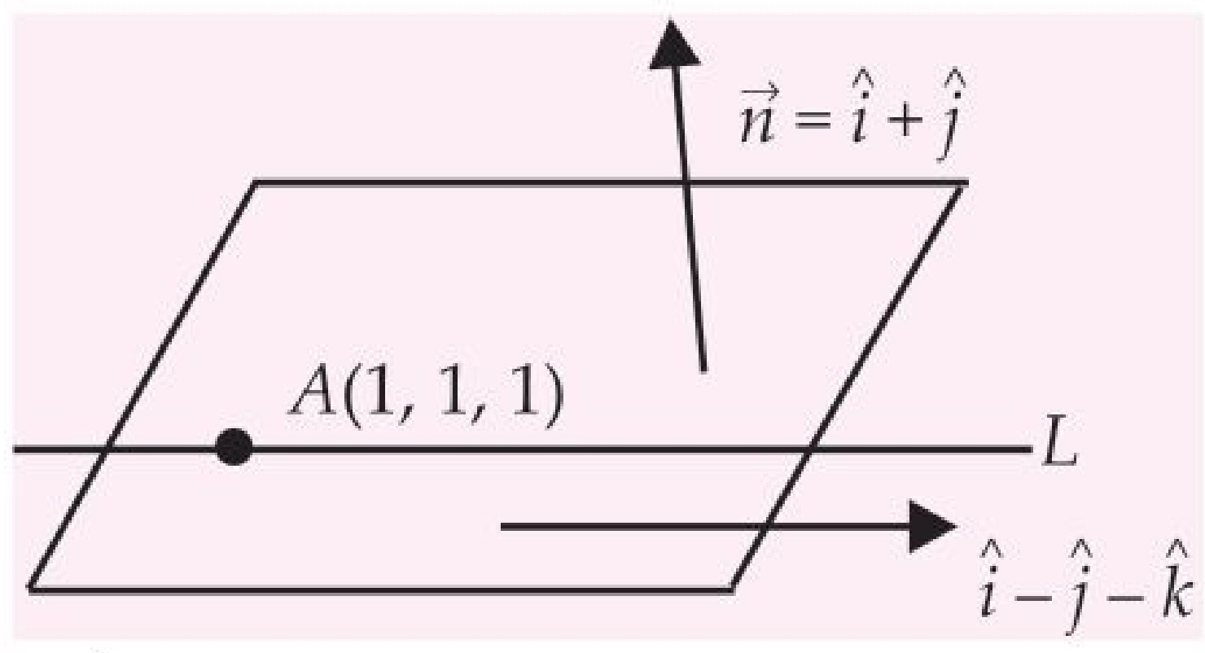
$$\int \operatorname{cosec}^2(x^2+y^2)\, d(x^2+y^2) = 2\int\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad -\cot(x^2+y^2) = \left(\frac{x}{y}\right)^2 + c,$$

which is the required solution.

8. (a, d) : $(\vec{r} - \vec{a})\cdot\vec{n}_1 = 0$; $\vec{r}\cdot\vec{n}_1 = \vec{a}\cdot\vec{n}_1$

$x + y = 2 \quad \Rightarrow$ Equation of P_1 is $x + y = 2$



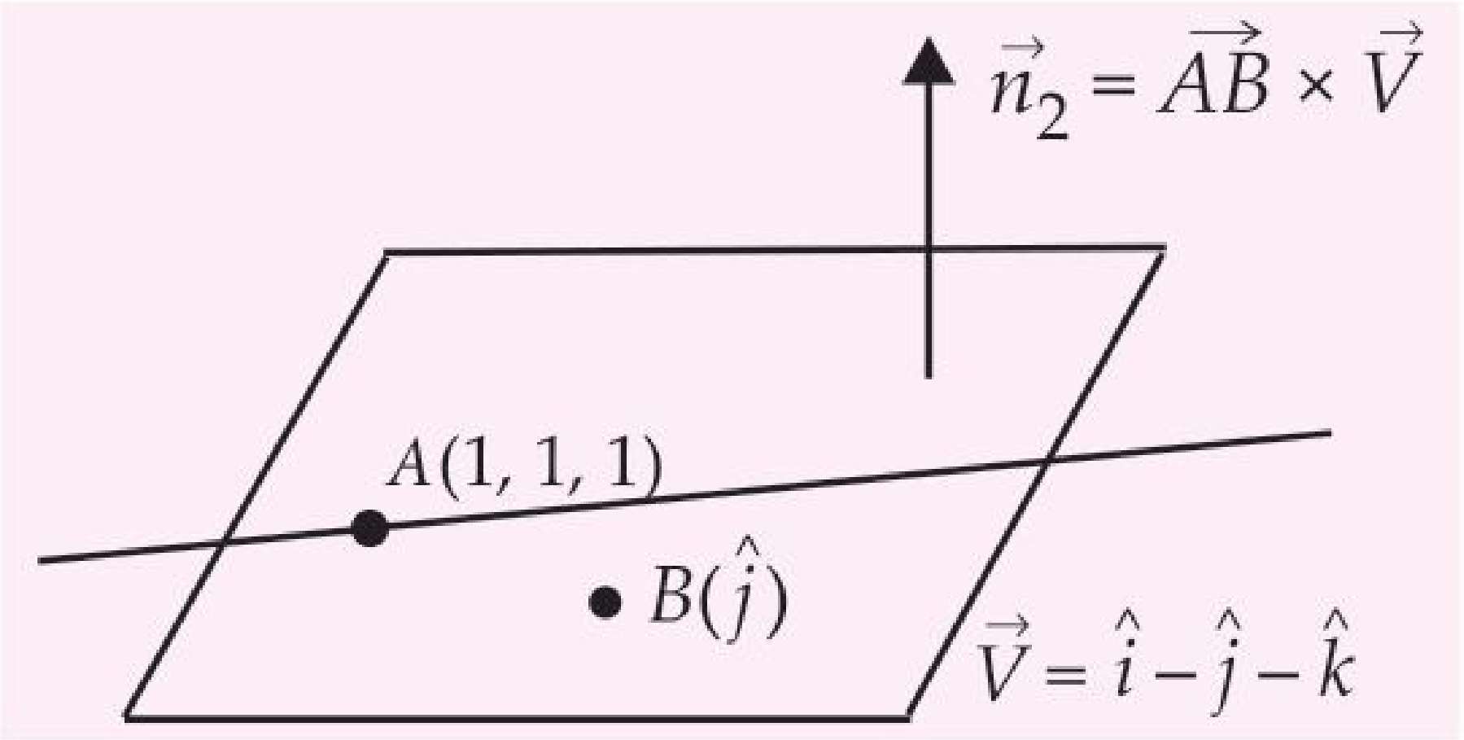
Now, $\overrightarrow{AB} = \hat{i} + \hat{k}$

and $\overrightarrow{AB} \times \vec{V} = \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix}$

$\therefore \quad \vec{n}_2 = \hat{i} + 2\hat{j} - \hat{k}$

$\therefore$ Equation of P_2 is $(\vec{r} - \hat{j})\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 0$

$\Rightarrow \quad \vec{r}\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 2$


$\vec{n}_2 = \overrightarrow{AB} \times \vec{V}$
A(1, 1, 1)
B($\hat{j}$)
$\vec{V} = \hat{i} - \hat{j} - \hat{k}$

If θ is acute angle between P_1 and P_2, we get

$$\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$$

As L is contained in P_1 and P_2 $\Rightarrow$ $\theta = 0$

9. (c, d) : Dr's of one of the bisectors are $(l_1 + l_2, m_1 + m_2, n_1 + n_2)$

Dc's are $\left(\frac{l_1+l_2}{r}, \frac{m_1+m_2}{r}, \frac{n_1+n_2}{r}\right)$, where

$$r = \sqrt{(l_1+l_2)^2 + (m_1+m_2)^2 + (n_1+n_2)^2} = 2\cos\frac{\theta}{2}$$

D.r.'s of another bisector are $(l_1 - l_2, m_1 - m_2, n_1 - n_2)$

D.c.'s of another bisector are

$\left(\frac{l_1-l_2}{r}, \frac{m_1-m_2}{r}, \frac{n_1-n_2}{r}\right)$, where

$$r=\sqrt{(l_1-l_2)^2+(m_1-m_2)^2+(n_1-n_2)^2}=2\sin\frac{\theta}{2}$$

10. (c, d) : Put $5(4x-5)=t^2 \Rightarrow 4x-5=\frac{t^2}{5}$

$$\Rightarrow x=\frac{t^2+25}{20} \quad \Rightarrow \quad dx=\frac{t}{10}dt$$

$$I=\int_{\sqrt{15}}^{\sqrt{35}}\left(\sqrt{\frac{t^2+25}{10}-t}+\sqrt{\frac{t^2+25}{10}+t}\right)\frac{t}{10}\,dt$$

$$=\frac{1}{10}\int_{\sqrt{15}}^{\sqrt{35}}\left(\frac{|t-5|}{\sqrt{10}}+\frac{|t+5|}{\sqrt{10}}\right)t\,dt$$

$$=\frac{1}{10}\int_{\sqrt{15}}^{5}\left(\frac{5-t+t+5}{\sqrt{10}}\right)tdt+\frac{1}{10}\int_{5}^{\sqrt{35}}\left(\frac{t-5+t+5}{\sqrt{10}}\right)tdt$$

$$=\frac{1}{\sqrt{10}}\left(\frac{t^2}{2}\right)_{\sqrt{15}}^{5}+\frac{1}{5\sqrt{10}}\left(\frac{t^3}{3}\right)_{5}^{\sqrt{35}}$$

$$=\frac{1}{\sqrt{10}}\left[\frac{25}{2}-\frac{15}{2}\right]+\frac{1}{15\sqrt{10}}(35\sqrt{35}-125)$$

$$=\frac{\sqrt{10}}{2}+\frac{7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}}$$

Clearly, given point lies in the given line, hence perpendicular distance is zero.

11. (a, c) : $\vec{v}=\vec{a}\times\vec{b}$

Let θ be the angle between $\vec{a}$ and $\vec{b}, |\vec{v}|=|\vec{a}||\vec{b}|\sin\theta=\sin\theta$

Now, $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}=\vec{a}-\vec{b}\cos\theta$

$$\Rightarrow \ |\vec{u}|^2=|\vec{a}|^2+|\vec{b}|^2\cos^2\theta-2(\vec{a}\cdot\vec{b})\cos\theta$$

$$=1+\cos^2\theta-2\cos^2\theta=1-\cos^2\theta=\sin^2\theta \ \Rightarrow |\vec{u}|=|\vec{v}|$$

Again, $\vec{u}\cdot\vec{b}=(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b})=0 \quad \Rightarrow |\vec{u}\cdot\vec{b}|=0$

12. (b, c) : $x^2+y^2+(4-x-y)^2=6$

$2x^2+2y^2-8x-8y+2xy+16=6$

$x^2+y^2-4x-4y+xy+5=0$

$y^2+(x-4)y+x^2-4x+5=0$

y being real $\Rightarrow \ (x-4)^2-4(x^2-4x+5)\geq 0$

$\Rightarrow \ 3x^2-8x+4\leq 0$

$\Rightarrow \ (3x-2)(x-2)\leq 0 \ \Rightarrow \ \frac{2}{3}\leq x\leq 2$

$x=\frac{2}{3} \ \Rightarrow \ y=z=\frac{5}{3}$ and $x=2 \ \Rightarrow \ y=z=1$

13. (2) : $\int\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right)dx=u(x)+c\Rightarrow\frac{f'(x)}{f(x)}=u(x)+c$

$$\frac{f'(x)}{f(x)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\ldots+\frac{1}{x-n}$$

$\Rightarrow \ u(x)=\frac{f'(x)}{f(x)}+C$, $u(x)=5$ has either $n-1$ or n roots.

$\therefore \ n=8$ or 9.

14. (1.16) : $f(x)=\min\left\{x+1, \ \sqrt{1-x}\right\}$

$$=\begin{cases}x+1, & -1\leq x\leq 0\\ \sqrt{1-x}, & 0<x\leq 1\end{cases}$$

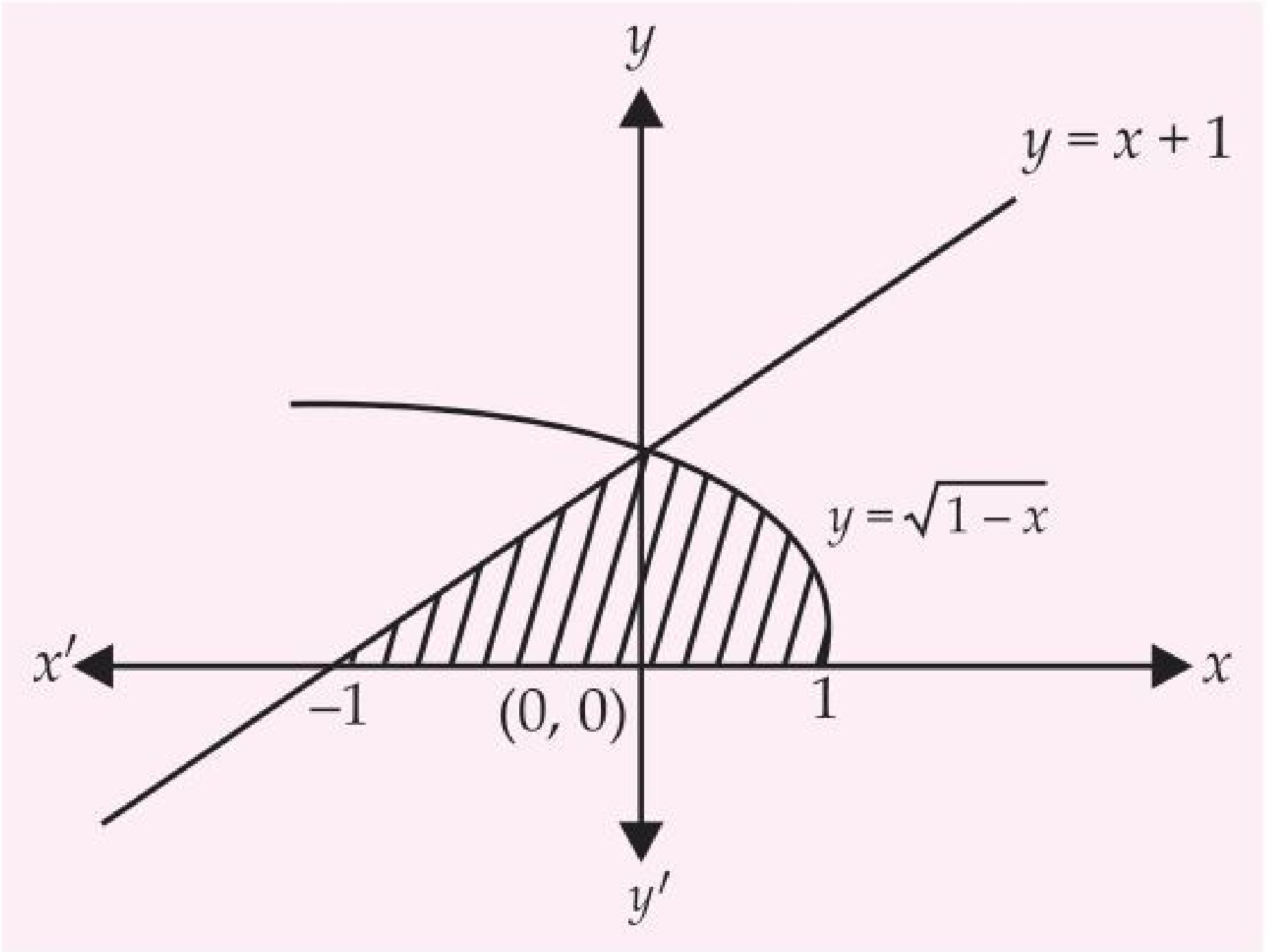
SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

		9			5			1					9	4	6					8
7	2					5	8					5					9		6	
			3				4							8				4		9
1	3	2	8			7		4				9			4			3		
4	8	5														3	1	9		7
													7		9		2	1		
			2		3		1				3									
				1	9							8	3	7	1		6		4	5
2	4	1		7						8		1	4		7		5		9	
							4	1				3	8	6						
								6	8	3		4		5						
						7					4			1						
				7		3	2		6	7	8	5							8	
	8		9			6	5	4			2				6				3	
		4			5				4		5		6					5		4
		9			2	7														1
		2	6	4								4		2		9		8		
8			5	9	7	4	1						8		4					5
	4			2	6		8					3		1		6			9	8
		8			4								4	6	8		7	2		
	6			3			4	7				8				3		7		

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.



$\therefore$ Required area $= \int_{-1}^{0}(x+1)dx + \int_{0}^{1}\sqrt{1-x}\,dx$

$= \frac{7}{6}$ sq. units

15. (670) : $f(-x)f'(x) - f(x)f'(-x) = 0$

$\Rightarrow \frac{d}{dx}(f(x)f(-x)) = 0$

$\Rightarrow f(x)f(-x) = K$, where K is arbitrary constant

$\Rightarrow f(x)f(-x) = 9 \qquad (\because f(0) = 3)$

$\Rightarrow f(-x) = \frac{9}{f(x)} \qquad \text{... (i)}$

Let $I = \int_{-2010}^{2010}\frac{dx}{3+f(x)} = \int_{-2010}^{2010}\frac{dx}{3+\frac{9}{f(-x)}}$ (Using (i))

$$= \frac{1}{3}\int_{-2010}^{2010}\frac{f(x)}{f(x)+3}dx$$

$$\Rightarrow 2I = \int_{-2010}^{2010}\left(\frac{1}{3+f(x)} + \frac{f(x)}{3(f(x)+3)}\right)dx = \frac{1}{3}\int_{-2010}^{2010}dx$$

$$\Rightarrow I = \frac{1}{6}(4020) = 670$$

16. (6.8) : Clearly, $x = \frac{\vec{d}\cdot\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}, y = \frac{\vec{d}\cdot\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}, z = \frac{\vec{d}\cdot\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

17. (6) : $f(x) = \begin{cases} x^2 - 5x + 6, & x \in [0, 2] \\ -(x^2 - 5x + 6), & x \in [2, 2.4] \end{cases}$

$f'(x) = \begin{cases} 2x - 5, & x \in [0, 2] \\ -2x + 5, & x \in [2, 2.4] \end{cases}$

$f'(x) = 0 \quad \Rightarrow \quad x = \frac{5}{2} \notin [0,\ 2.4]$

Further, $f'(x) < 0$ for $0 < x < 2$ and $f'(x) > 0$ for $2 < x < 2.4$

$\therefore$ f is minimum at $x = 2$ and maximum at $x = 0$.

Here, $f(0) = 6$ is maximum because $f(2.4) = 0.24$ and $f(2) = 0$

18. (2) : Let $S_n = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}$

$= 2 + \left(\frac{-1}{a_1} + \frac{1}{a_2}\right) + \frac{1}{a_3} + ... + \frac{1}{a_n}$

$= 2 + \left(\frac{a_1 - a_2}{a_1a_2}\right) + \frac{1}{a_3} + ... + \frac{1}{a_n}$

$= 2 + \left(\frac{-1}{a_1a_2} + \frac{1}{a_3}\right) + \frac{1}{a_4} + ... + \frac{1}{a_n}$

$= 2 + \left(\frac{a_1a_2 - a_3}{a_1a_2a_3}\right) + \frac{1}{a_4} + ... + \frac{1}{a_n}$

$= 2 + \left(\frac{-1}{a_1a_2a_3} + \frac{1}{a_4}\right) + ... + \frac{1}{a_n}$

= ....

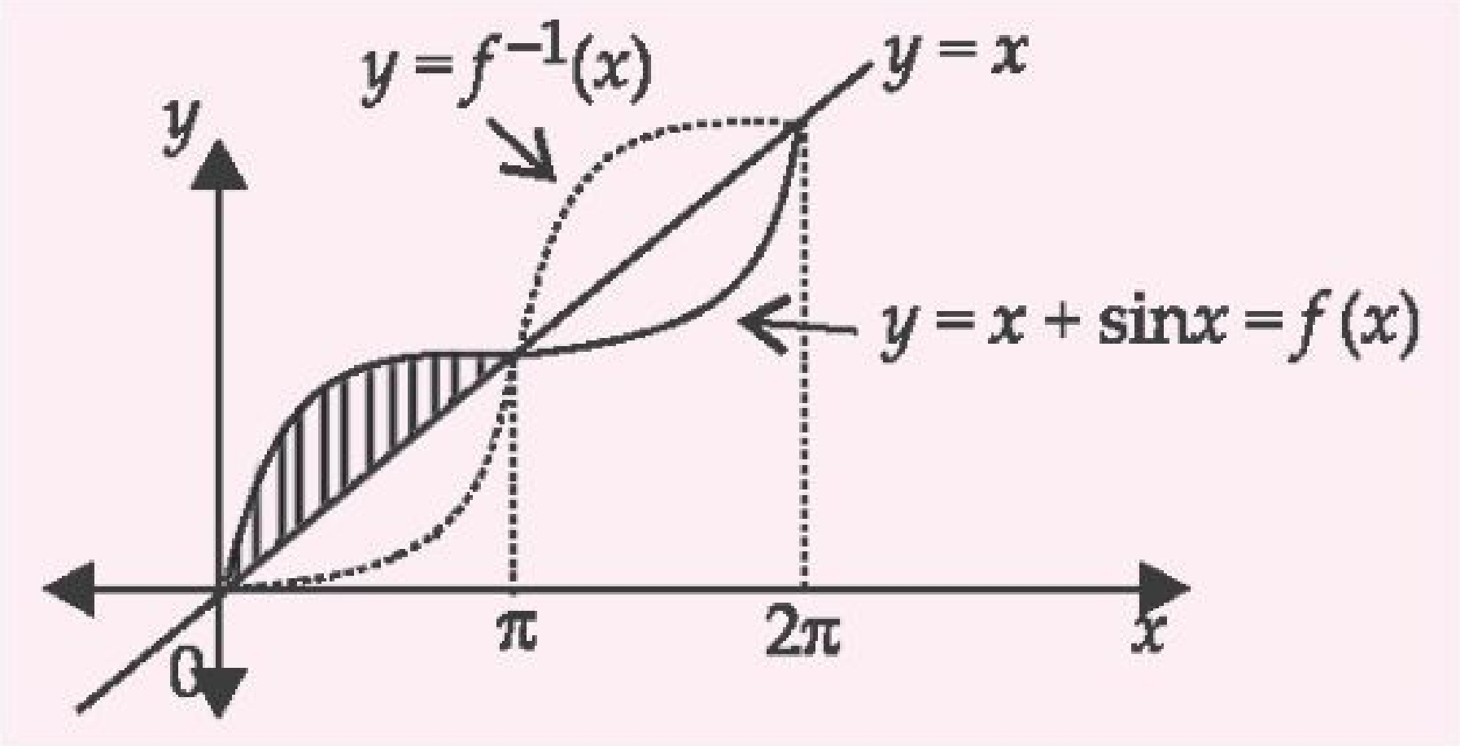
Finally, $S_n = 2 - \frac{1}{a_1a_2a_3....a_n} \qquad \therefore \lim_{n\to\infty} S_n = 2$

PAPER-2

1. (8) : Area of shaded part is

$$A = \int_0^{\pi}(x + \sin x)dx - \int_0^{\pi}x\,dx = \left[\frac{x^2}{2} - \cos x\right]_0^{\pi} - \left[\frac{x^2}{2}\right]_0^{\pi} = 2$$

Required area = $4A$ = 8 sq. units



2. (1) : Clearly, $(f'(x) - 1)^2 \geq 0 \Rightarrow 0 \leq \int_0^1 (f'(x) - 1)^2 dx$

$$= \int_0^1 (f'(x))^2 dx - 2\int_0^1 f'(x)dx + \int_0^1 dx$$

$$= \int_0^1 (f'(x))^2 dx - 2[f(1) - f(0)] + 1$$

$$= \int_0^1 (f'(x))^2 dx - 2 + 1$$

Minimum value of $\int_0^1 (f'(x))^2 dx$ is 1 which will be corresponding to the function $f(x) = x \ \forall \ x \in [0,\ 1]$.

3. (5) : According to the given condition is

$$\frac{dy}{dt}=-k\sqrt{y} \Rightarrow \int_4^0 \frac{dy}{\sqrt{y}}=-k\int_0^t dt$$

$$\left[2\sqrt{y}\right]_4^0=-kt=\frac{-t}{15} \Rightarrow \frac{t}{15}=4 \Rightarrow t=60 \text{ min}$$

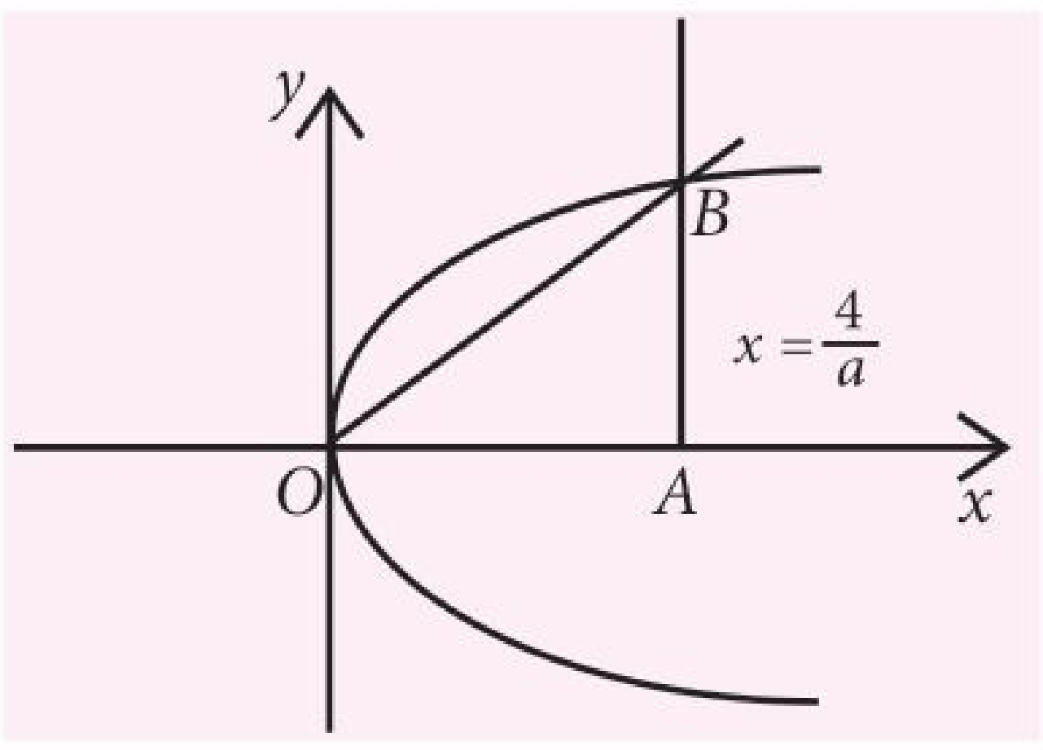
4. (1) : The given equation can be written as
$3(1+x)^2+3(1-y)^2+4(1+x)(1-y)=0$

$\Rightarrow (1+x)^2+(1-y)^2+2[(1+x)+(1-y)]^2=0$

$\therefore\ x=-1, y=1$

5. (2) : Given $\frac{1}{3}=\int_0^{4/a} 2\sqrt{a}\cdot\sqrt{x}dx-\text{area of }\Delta OAB$

$\Rightarrow a=8 \Rightarrow x^2+2x-8=0 \Rightarrow x=-4, 2$



6. (0) : $\int\frac{1}{(x^3-8)}dx=\frac{x}{(x^3-8)}-\int\frac{-3x^2}{(x^3-8)}xdx$

7. (a, d) : Given $a+b=2$

and $\int_0^4 (a\sqrt{x}+bx)dx=8 \Rightarrow 2a+3b=3$

$\therefore\ a=3, b=-1$

8. (a, b) : $\frac{y^2dx-2xydy}{y^4}=\frac{x^3y^3dy+x^2y^4dx}{y^4}$

$$\Rightarrow d\left(\frac{x}{y^2}\right)=\frac{x^2y^3(xdy+ydx)}{y^4}$$

$$\Rightarrow d\left(\frac{x}{y^2}\right)=\frac{x^2}{y}d(xy) \Rightarrow d\left(\frac{x}{y^2}\right)=\left(\frac{x}{y^2}\right)xy\,d(xy)$$

$$\Rightarrow \frac{d\left(\frac{x}{y^2}\right)}{\left(\frac{x}{y^2}\right)}=xyd(xy) \Rightarrow \log\left(\frac{x}{y^2}\right)=\frac{(xy)^2}{2}+c$$

9. (a, d) : we have $\vec{x}+\vec{y}+\vec{z}=\vec{a}$ … (i)

Taking dot product with $\vec{x}, \vec{y}$ respectively, we get

$1+\vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{z}=\vec{a}\cdot\vec{x} \Rightarrow (\vec{y}+\vec{z})\cdot\vec{x}=\frac{1}{2}$ … (ii)

$1+(\vec{x}+\vec{z})\cdot\vec{y}=\frac{7}{4} \Rightarrow (\vec{x}+\vec{z})\cdot\vec{y}=\frac{3}{4}$ … (iii)

From (i),

$(\vec{x}+\vec{y}+\vec{z})^2=|\vec{a}|^2 \Rightarrow 1+1+1+2(\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{z}+\vec{z}\cdot\vec{x})=4$

$\Rightarrow \vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{z}+\vec{z}\cdot\vec{x}=\frac{1}{2}$ … (iv)

From (ii), (iii) and (iv) we get

$$\vec{y}\cdot\vec{z}=0,\ \vec{x}\cdot\vec{y}=\frac{3}{4},\ \vec{z}\cdot\vec{x}=-\frac{1}{4}$$

Also, we have, $(\vec{x}\cdot\vec{z})\vec{y}-(\vec{x}\cdot\vec{y})\vec{z}=\vec{b}$ (given)

$\Rightarrow \frac{-1}{4}\vec{y}-\frac{3}{4}\vec{z}=\vec{b}$ … (v)

and $(\vec{z}\cdot\vec{x})\vec{y}-(\vec{z}\cdot\vec{y})\vec{x}=\vec{c}$

$\Rightarrow \vec{y}=-4\vec{c}$ and $\vec{z}=\frac{4}{3}(\vec{c}-\vec{b}) \Rightarrow \vec{x}=\frac{1}{3}(3\vec{a}+4\vec{b}+8\vec{c})$

10. (a, d) : Let A = centre of $C_1=(0,0)$ and B = centre of $C_2=(3,4)$

Also, let r_1 = radius of $C_1=2$ and r_2 = radius of $C_2=7$

$\therefore\ r_2-r_1=5=AB$

$\therefore$ Touch each other internally.

11. (b, c, d) : $I_n=\int_{-\pi}^{\pi}\frac{\sin(nx)}{(1+e^x)\sin x}dx$

$$=\int_{-\pi}^{\pi}\frac{\sin((-\pi+\pi-x)n)}{(1+e^{-\pi+\pi-x})\sin(-\pi+\pi-x)}dx$$

$$=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+e^{-x})\sin x}dx$$

$$\therefore\ 2I_n=\int_{-\pi}^{\pi}\left(\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\right)\frac{\sin nx}{\sin x}dx=\int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx$$

$$=2\int_0^{\pi}\frac{\sin nx}{\sin x}dx \quad\therefore\quad I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

$$I_{n+2}-I_n=\int_0^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx$$

$$=\int_0^{\pi}2\cos(n+1)xdx=\frac{2}{n+1}[\sin(n+1)x]_0^{\pi}=0$$

$\therefore\ I_{n+2}=I_n$

$\therefore\ I_0=I_2=I_4=....=I_{2k}=0$ and $I_1=I_3=I_5=...=I_{2k+1}=\pi$

because $I_1=\int_0^{\pi}\frac{\sin x}{\sin x}dx=\pi$

12. (b, c) : Clearly the equal roots of the given equation are 1, 1

$$\therefore \quad \frac{c(a-b)}{a(b-c)}=1 \;\Rightarrow\; ca-bc=ab-ca \;\Rightarrow\; b=\frac{2ca}{c+a}$$

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in H.P.}$$

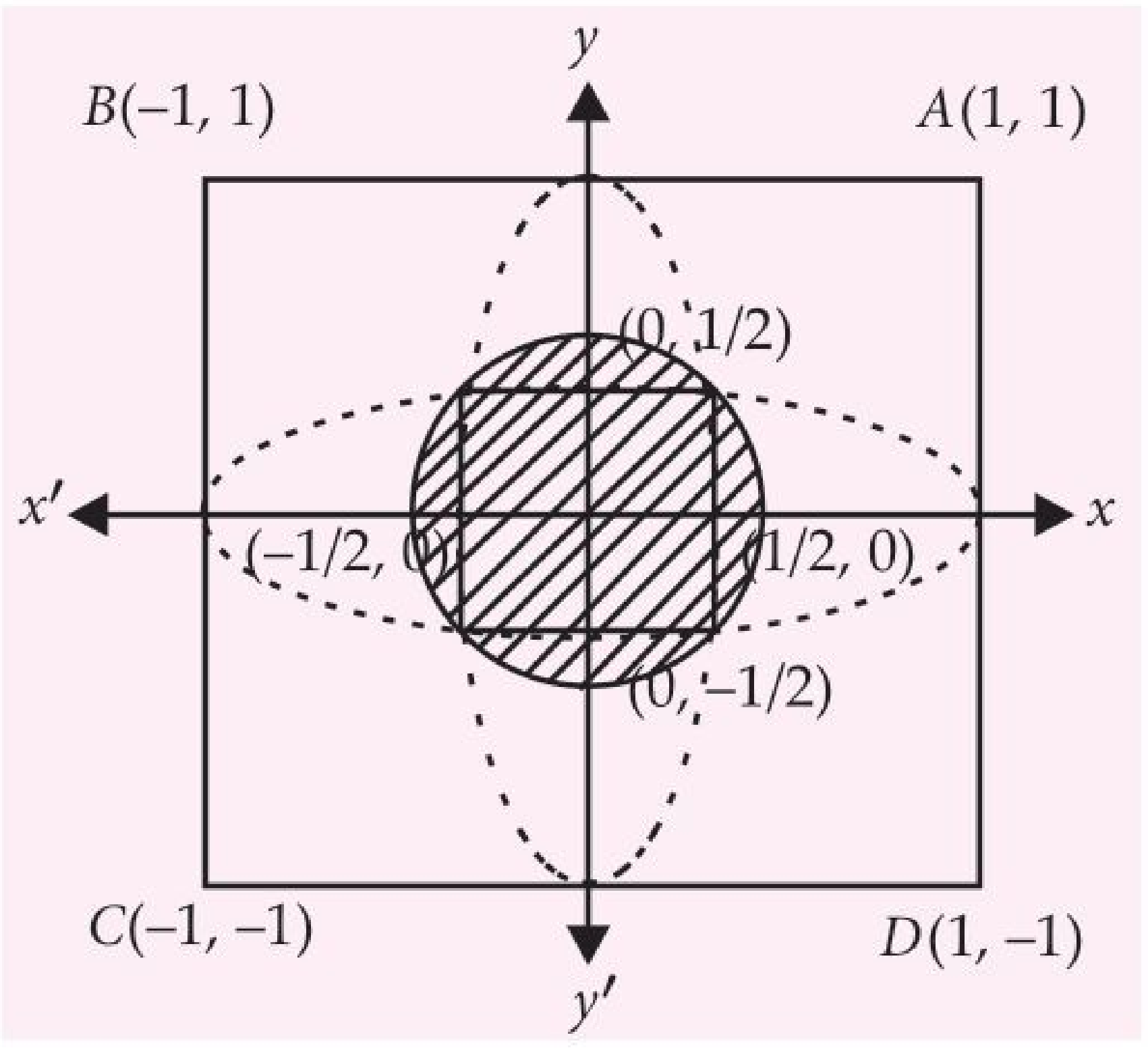
$$\Leftrightarrow \quad \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \text{ are in A.P.}$$

$$\Leftrightarrow \quad \frac{b+c}{a}+1, \frac{c+a}{b}+1, \frac{a+b}{c}+1 \text{ are in A.P.}$$

$$\Leftrightarrow \quad \frac{a+b+c}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c} \text{ are in A.P.}$$

$$\Leftrightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P. which is true.}$$

13. (c) :



The equation of the sides of the square are as follows:

$AB : y = 1, BC : x = -1, CD : y = -1, DA : x = 1$

Let (x, y) be any point inside s.

Then according to given conditions

$$\sqrt{x^2+y^2} \le |1-x|, |1+x|, |1-y|, |1+y|$$

$$\Rightarrow \; x^2 + y^2 < (1 - x)^2, (1 + x)^2, (1 - y)^2, (1 + y)^2$$

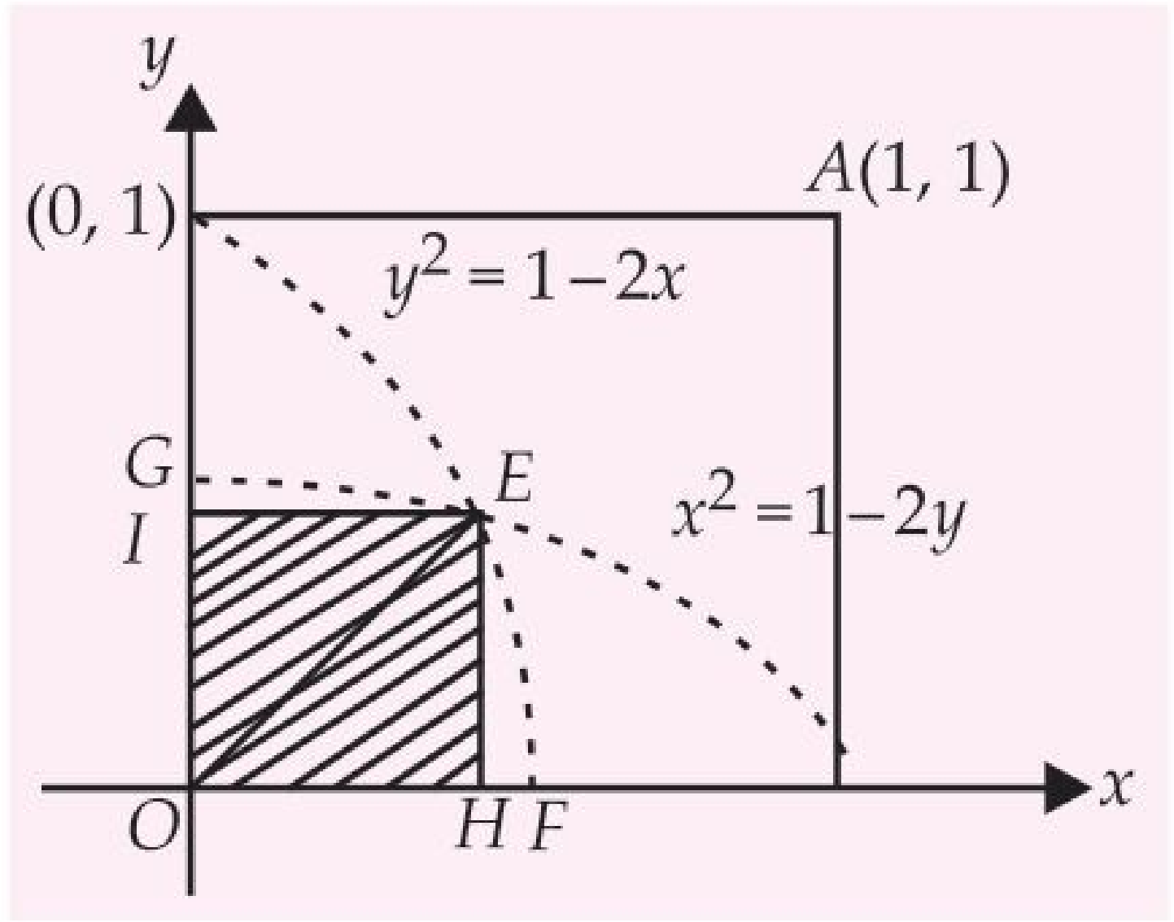
$$\Rightarrow \; x^2 + y^2 < x^2 - 2x + 1, x^2 + 2x + 1, y^2 - 2y + 1, y^2 + 2y + 1$$

$$\Rightarrow \; y^2 < 1 - 2x, y^2 < 1 + 2x, x^2 < 1 - 2y \text{ and } x^2 < 2y + 1$$

Therefore, the region S is the region lying inside the four parabolas.

$y^2 = 1 - 2x, y^2 = 1 + 2x, x^2 = 1 + 2y, x^2 = 1 - 2y$

14. (b) :



Clearly the required point is E.

Now, $y^2 = 1 - 2x$ and $x^2 = 1 - 2y$ is intersect on the line $y = x$

The point of intersection of these, *i.e.*, $E=(\sqrt{2}-1, \sqrt{2}-1)$

15. (b) : Since, S is symmetrical in all quadrants

Therefore, $S = 4 \times$ area lying in the first quadrant

Area of the region $OEFO$

= area of ΔOEH + area of $HEFH$

$$=\frac{1}{2}(\sqrt{2}-1)^2+\int_{\sqrt{2}-1}^{1/2}\sqrt{1-2x}\,dx$$

$$=\frac{1}{2}(\sqrt{2}-1)^2+\left[(1-2x)^{3/2}\frac{2}{3}\cdot\frac{1}{2}(-1)\right]_{\sqrt{2}-1}^{1/2}$$

$$=\frac{1}{2}(3-2\sqrt{2})+\frac{1}{3}[(\sqrt{2}-1)^2]^{3/2}$$

$$=\frac{1}{2}(3-2\sqrt{2})+\frac{1}{3}[5\sqrt{2}-7]$$

$$=\frac{1}{6}[9-6\sqrt{2}+10\sqrt{2}-14]=\frac{1}{6}[4\sqrt{2}-5]$$

Similarly, area $OEGO=\frac{1}{6}(4\sqrt{2}-5)$

Thus, area lying in the first quadrant

$$=\frac{2}{6}(4\sqrt{2}-5)=\frac{1}{3}(4\sqrt{2}-5)$$

Hence, $S=\frac{4}{3}(4\sqrt{2}-5)=\frac{1}{3}(16\sqrt{2}-20)$

16. (d) **17. (b)**

18. (c) : $f(0) = f(0 + 0) = f(0)\,f(0)$

$\Rightarrow \; f(0) = 1 \;(\because f(x) \ne 0 \;\forall x)$

$$f'(x)=\lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)f(h)-f(x))}{h}=\lim_{h\to 0}f(x)\frac{(f(h)-f(0))}{h}$$

$(\because f(0) = 1)$

$$=f(x)f'(0)=2f(x)$$

$$\therefore \; \frac{f'(x)}{f(x)}=2 \;\Rightarrow\; \log|f(x)|=2x+c$$

$\Rightarrow \; \log(1) = 0 + c$ for $x = 0 \;\Rightarrow\; c = 0$

$\therefore \; \log|f(x)| = 2x \;\Rightarrow\; \log|f(2)| = 4$

⁂

MONTHLY TEST DRIVE CLASS XI ANSWER KEY

1.	(c)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(c)
6.	(a)	**7.**	(a,b)	**8.**	(a,c)	**9.**	(b,c)	**10.**	(a,b,c)
11.	(b)	**12.**	(a,b,c)	**13.**	(a,c)	**14.**	(a)	**15.**	(b)
16.	(c)	**17.**	(4)	**18.**	(7)	**19.**	(5)	**20.**	(5)

FOCUS JEE PRACTICE QUESTIONS

MULTIPLE CHOICE QUESTIONS

1. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals

(a) 0 (b) 1 (c) 2 (d) ∞

2. $\lim\limits_{x\to\pi/2} \dfrac{[1-\tan(x/2)][1-\sin x]}{[1+\tan(x/2)][\pi-2x]^3} =$

(a) 0 (b) 1/32 (c) ∞ (d) 1/8

3. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (d) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$

4. The range of the function $F(x) = {}^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$

(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

5. If $0 < x < \pi$ and $\cos x + \sin x = 1/2$, then $\tan x$ is

(a) $\dfrac{(1-\sqrt{7})}{4}$ (b) $\dfrac{(4-\sqrt{7})}{3}$

(c) $-\dfrac{(4+\sqrt{7})}{3}$ (d) $\dfrac{(1+\sqrt{7})}{4}$

6. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex further, assume that the origin, z_1 and z_2 form an equilateral triangle, then

(a) $a^2 = 2b$ (b) $a^2 = 3b$

(c) $a^2 = 4b$ (d) $a^2 = b$

7. The integral $\displaystyle\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx$ equals

(a) $\dfrac{2\pi}{3} - 4 - 4\sqrt{3}$ (b) $4\sqrt{3} - 4$

(c) $4\sqrt{3} - 4 - \dfrac{\pi}{3}$ (d) $\pi - 4$

8. The differential equation of all circles passing through the origin and having their centres on the x-axis is

(a) $y^2 = x^2 + 2xy\dfrac{dy}{dx}$ (b) $y^2 = x^2 - 2xy\dfrac{dy}{dx}$

(c) $x^2 = y^2 + xy\dfrac{dy}{dx}$ (d) $x^2 = y^2 + 3xy\dfrac{dy}{dx}$

9. In a triangle ABC, if $A = \dfrac{\pi}{7}, B = \dfrac{2\pi}{7}, C = \dfrac{4\pi}{7}$, then $a^2 + b^2 + c^2$ is (R is circumradius of ΔABC)

(a) $4R^2$ (b) $6R^2$

(c) $7R^2$ (d) $8R^2$

10. Range of $f(x) = \sin^6 x + \cos^6 x$ is

(a) $[0, 1]$ (b) $[0, \sqrt{2}]$

(c) $\left[\dfrac{1}{\sqrt{2}}, \dfrac{3}{4}\right]$ (d) $\left[\dfrac{1}{4}, 1\right]$

11. Let $f(x) = x^2 + 5x + 6$, then the number of real roots of $(f(x))^2 + 5f(x) + 6 - x = 0$ is

(a) 1 (b) 2

(c) 3 (d) 0

12. Let $f(x)$ be a function such that $f(x) = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. Then the number of solutions of the equation $f(x)+f\left(\frac{1}{x}\right)=1$ is (are)

(a) 0 (b) 1 (c) 2 (d) infinite

13. If z_1, z_2, z_3 and z_4 be the consecutive vertices of a square, then $z_1^2+z_2^2+z_3^2+z_4^2$ equals

(a) $z_1z_2 + z_2z_3 + z_3z_4 + z_4z_1$

(b) $z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4$

(c) 0 (d) none of these

14. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2 is

(a) $16x^2 + 10xy + y^2 = 2$ (b) $16x^2 - 10xy + y^2 = 2$

(c) $16x^2 + 10xy + y^2 = 4$ (d) $16x^2 - 10xy + y^2 = 4$

15. If the tangent drawn at $P\left(t=\frac{\pi}{4}\right)$ to the curve $x = \sec^2 t$, $y = \cot t$ meets the curve again at R, then $PR =$

(a) $\frac{3\sqrt{5}}{2}$ (b) $\frac{2\sqrt{5}}{3}$ (c) $\frac{5\sqrt{5}}{4}$ (d) $\frac{4\sqrt{5}}{5}$

16. The maximum area of a rectangle whose two consecutive vertices lie on the x-axis and another two lie on the curve $y = e^{-|x|}$ is equal to

(a) $2e$ sq. units (b) $2/e$ sq. units

(c) e sq. units (d) $1/e$ sq. units

17. $\int_0^{4/\pi}\left(3x^2\cdot\sin\frac{1}{x}-x\cdot\cos\frac{1}{x}\right)dx=$

(a) $\frac{8\sqrt{2}}{\pi^3}$ (b) $\frac{24\sqrt{2}}{\pi^3}$ (c) $\frac{32\sqrt{2}}{\pi^3}$ (d) $\frac{32\sqrt{2}}{\pi}$

18. The general solution of the differential equation $\frac{dy}{dx}=y\tan x-y^2\sec x$ is

(a) $\tan x = (c + \sec x)y$ (b) $\sec y = (c + \tan y)x$

(c) $\sec x = (c + \tan x)y$ (d) none of these

19. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix}1 & -x\\ -x & 1\end{bmatrix}$ and $z=\frac{x+y}{1+xy}$. Then,

(a) $A(z) = A(x) + A(y)$ (b) $A(z) = A(x)\,[A(y)]^{-1}$

(c) $A(z) = A(x)\cdot A(y)$ (d) $A(z) = A(x) - A(y)$

20. A and B are two non singular matrices so that $A^6 = I$ and $AB^2 = BA$ $(B \neq I)$. A value of K so that $B^K = I$ is

(a) 31 (b) 32 (c) 63 (d) 64

NUMERICAL VALUE TYPE

21. Let $A=\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}$. If $|A^2| = 25$, then $|\alpha|$ equals_____.

22. If $f(x + y) = f(x)\,.\,f(y)$ $\forall$ x, y and $f(5) = 2$, $f'(0) = 3$, then the value of $f'(5)$ is ______.

23. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these triangles is ______.

24. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is ______.

25. A random variable X has the probability distribution:

X :	1	2	3	4	5	6	7	8
$P(X)$:	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X$ is a prime number$\}$ and $F = \{X < 4\}$, $P(E \cup F)$ is ______.

26. If $I_n=\int_0^{\pi/4}\tan^n x\,dx$, then $I_8+I_6=\frac{1}{\lambda}$, where λ equals ______.

27. The graphs of $f(x) = x^2$ and $g(x) = cx^3$ intersect at two points. If the area of the region bounded between $f(x)$ and $g(x)$ over the interval $\left(0,\frac{1}{c}\right)$ equal to $\frac{2}{3}$, then the value of $\frac{1}{c}+\frac{1}{c^2}$ is ______.

28. If $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$, $y(1)=1$, then one of the values of x_0 satisfying $y(x_0) = e$ is given by $\sqrt{\lambda}\,e$, where λ equals______.

29. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is ______.

30. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is ______.

SOLUTIONS

1. **(b) :** We have, $|z-1| = |z+1|$

$\Rightarrow |z-1|^2 = |z+1|^2 \Rightarrow (z-1)(\bar{z}-1) = (z+1)(\bar{z}+1)$

$\Rightarrow z\bar{z} - z - \bar{z} + 1 = z\bar{z} + z + \bar{z} + 1$

$\Rightarrow z + \bar{z} = 0$ (z being purely imaginary)

Thus, $x = 0$

Again, $|z-1|^2 = |z-i|^2 \Rightarrow (x-1)^2 + y^2 = x^2 + (y-1)^2$

$\Rightarrow 1 + y^2 = (y-1)^2$ $(\because x = 0)$

$\Rightarrow 1 + y^2 = y^2 - 2y + 1$

$\therefore y = 0$

Thus, (0, 0) is the desired point.

2. **(b) :** $\displaystyle\lim_{x\to\pi/2} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{4\cdot\left(\frac{\pi-2x}{4}\right)(\pi-2x)^2}$

$$= \lim_{x\to\pi/2} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4\cdot\left(\frac{\pi}{4}-\frac{x}{2}\right)} \frac{1-\cos\left(\frac{\pi}{2}-x\right)}{(\pi-2x)^2}$$

$$= \lim_{x\to\pi/2} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4\cdot\left(\frac{\pi}{4}-\frac{x}{2}\right)} \frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4^2\left(\frac{\pi-2x}{4}\right)^2} = \frac{1}{4}\times\frac{2}{16} = \frac{1}{32}$$

3. **(d) :** $f'(x) = \dfrac{1}{1+(\sin x+\cos x)^2}.(\cos x - \sin x)$

$$\Rightarrow f'(x) = \frac{\cos x - \sin x}{2+\sin 2x}$$

If $f'(x) > 0$ then $f(x)$ is an increasing function.

For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{4}, \cos x > \sin x$

Hence, $y = f(x)$ is increasing in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

4. **(c) :** $F(x)$ to be defined for $x \in N$.

(i) $\therefore 7 - x > 0 \Rightarrow x < 7$

(ii) $x - 3 \geq 0 \Rightarrow x \geq 3$

(iii) $x - 3 \leq 7 - x \Rightarrow x \leq 5$

$\therefore$ From (i), (ii) and (iii), $x = 3, 4, 5$

$\therefore F(3) = {}^4P_0, F(4) = {}^3P_1, F(5) = {}^2P_2$

$\therefore$ {1, 2, 3} is required range.

5. **(c) :** Given, $\cos x + \sin x = \dfrac{1}{2}$

$\Rightarrow 1 + \sin 2x = \dfrac{1}{4}$ (By squaring both sides)

$$\Rightarrow \frac{2\tan x}{1+\tan^2 x} = \frac{-3}{4}$$

$\Rightarrow 3\tan^2 x + 8\tan x + 3 = 0$

$$\therefore \tan x = \frac{-8\pm\sqrt{64-36}}{6} = \frac{-4\pm\sqrt{7}}{3}$$

6. **(b) :** As z_1, z_2 are roots of $z^2 + az + b = 0$

$\therefore z_1 + z_2 = -a, z_1z_2 = b$

Again 0, z_1, z_2 are vertices of an equilateral triangle

$\therefore 0^2 + z_1^2 + z_2^2 = 0z_1 + z_1z_2 + z_20 = 0$

$\Rightarrow z_1^2 + z_2^2 = z_1z_2 \Rightarrow (z_1+z_2)^2 = 3z_1z_2$

$\Rightarrow a^2 = 3b$

7. **(c) :** Let $I = \displaystyle\int_0^\pi \sqrt{1+4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx$

$$= \int_0^\pi \sqrt{\left(1-2\sin\frac{x}{2}\right)^2}\,dx = \int_0^\pi \left|1-2\sin\frac{x}{2}\right| dx$$

$$= \int_0^{\pi/3} \left(1-2\sin\frac{x}{2}\right)dx + \int_{\pi/3}^{\pi} \left(2\sin\frac{x}{2}-1\right)dx$$

$$= \left[\left(x+4\cos\frac{x}{2}\right)\right]_0^{\pi/3} + \left[\left(-4\cos\frac{x}{2}-x\right)\right]_{\pi/3}^{\pi}$$

$$= -\frac{\pi}{3} + 8\cdot\frac{\sqrt{3}}{2} - 4 = 4\sqrt{3} - 4 - \frac{\pi}{3}$$

8. **(a) :** General equation of all such circles is

$(x-h)^2 + (y-0)^2 = h^2$ ....(i) where h is parameter

$\Rightarrow (x-h)^2 + y^2 = h^2$

Differentiating w.r.t. x, we get

$$2(x-h) + 2y\frac{dy}{dx} = 0 \Rightarrow h = x + y\frac{dy}{dx}$$

Putting the value of h in (i), we get

$$y^2 = x^2 + 2xy\frac{dy}{dx}$$

9. **(c) :** $a^2 + b^2 + c^2 = 4R^2(\sin^2A + \sin^2B + \sin^2C)$

$= 2R^2(1 - \cos2A + 1 - \cos2B + 1 - \cos2C)$

$$= 2R^2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)\right]$$

$$= 2R^2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)\right]$$

$$= 2R^2\left[3 - \left(2\sin\frac{\pi}{7}\cdot\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cdot\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cdot\cos\frac{6\pi}{7}\right)\frac{1}{2\sin\frac{\pi}{7}}\right]$$

$$=2R^2\left[3-\frac{1}{2\sin\frac{\pi}{7}}\left(\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}+\sin\frac{7\pi}{7}-\sin\frac{5\pi}{7}\right)\right]$$

$$=2R^2\left[3+\frac{1}{2}\right]=7R^2$$

10. (d) : $f(x) = (\sin^2x + \cos^2x)^3 - 3\sin^2x\cos^2x(\sin^2x + \cos^2x)$

$$=1-\frac{3}{4}\sin^2 2x$$

$\because$ Range of $\sin^2 2x$ is $[0, 1]$

$\therefore$ Range of $f(x)$ is $\left[\frac{1}{4},1\right]$.

11. (d) : Use "$f(x) = x$ has non real roots $\Rightarrow f(f(x)) = x$ also has non-real roots".

12. (d) : Given, $f(x) = x - [x], x \in R - \{0\}$

Now, $f(x)+f\left(\frac{1}{x}\right)=1 \Rightarrow x-[x]+\frac{1}{x}-\left[\frac{1}{x}\right]=1$

$$\Rightarrow \left(x+\frac{1}{x}\right)-\left([x]+\left[\frac{1}{x}\right]\right)=1$$

$$\Rightarrow \left(x+\frac{1}{x}\right)=[x]+\left[\frac{1}{x}\right]+1 \qquad \text{...(i)}$$

Clearly, R.H.S. is an integer $\therefore$ L. H. S. is also an integer.

Let $x+\frac{1}{x}=k$ (an integer) $\Rightarrow x^2-kx+1=0$

$$\therefore \quad x=\frac{k\pm\sqrt{k^2-4}}{2}$$

For real values of x, $k^2 - 4 \geq 0 \Rightarrow k \geq 2$ or $k \leq -2$

Also, $k = 2$ and -2 does not satisfy equation (i)

$\therefore$ The equation (i) will have solutions if $k > 2$ or $k < -2$, $k \in Z$.

Hence, equation (i) has infinite number of solutions.

13. (a) : We know that, $\frac{z_2-z_1}{z_4-z_1}=\left|\frac{z_2-z_1}{z_4-z_1}\right|e^{\frac{i\pi}{2}}=i$

$[\because |z_2 - z_1| = |z_4 - z_1|]$

$$\Rightarrow (z_4 - z_1)^2 + (z_2 - z_1)^2 = 0 \qquad \text{...(i)}$$

Similarly, $\frac{z_4-z_3}{z_2-z_3}=i$

$$\Rightarrow (z_4 - z_3)^2 + (z_2 - z_3)^2 = 0 \qquad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2[z_1^2+z_2^2+z_3^2+z_4^2-z_1z_2-z_4z_1-z_4z_3-z_2z_3]=0$$

$$\Rightarrow z_1^2+z_2^2+z_3^2+z_4^2=z_1z_2+z_2z_3+z_3z_4+z_4z_1$$

14. (a) : Let $P(h, k)$ be the required point.

$$\therefore \quad y - k = 4(x - h) \qquad \text{...(i)}$$

Let it meets $xy = 1$ ...(ii) at $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\Rightarrow \quad x_1+x_2=\frac{4h-k}{4}, x_1x_2=-\frac{1}{4}$$

Also, $\frac{2x_1+x_2}{3}=h \Rightarrow x_1=\frac{8h+k}{4}, x_2=-\frac{2h+k}{2}$

$\therefore$ Required locus is $16x^2 + 10xy + y^2 = 2$.

15. (a) : We have, $x = \sec^2 t, y = \cot t$ ...(i)

At $t=\frac{\pi}{4}$, $x=2$, $y=1 \Rightarrow P$ is $(2, 1)$

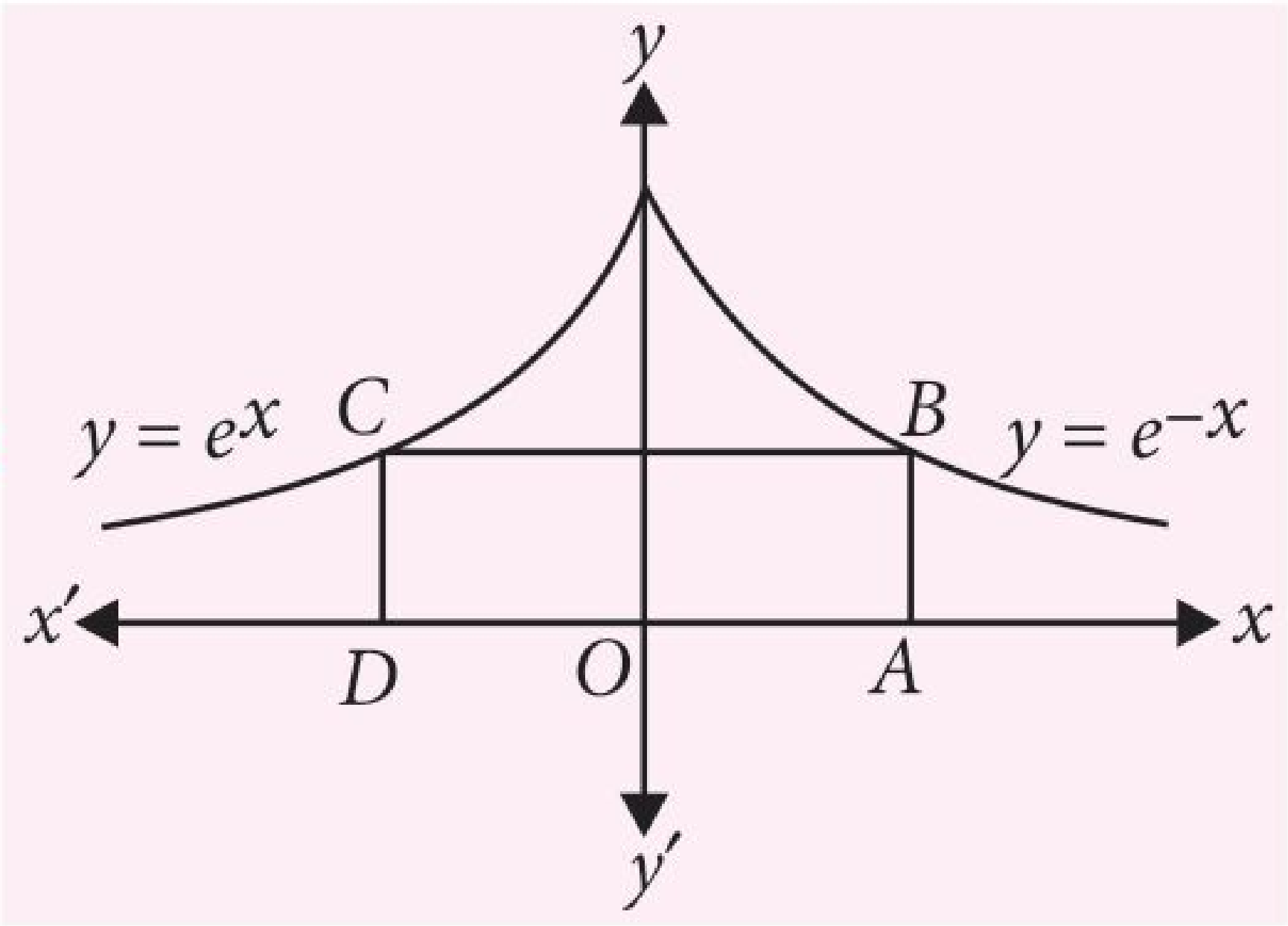
$$\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}}=\left[\frac{-\text{cosec}^2 t}{2\sec t\cdot\sec t\cdot\tan t}\right]_{t=\frac{\pi}{4}}=-\frac{1}{2}$$

$\therefore$ Tangent at $P(2, 1)$ is, $y=\frac{4-x}{2}$ ...(ii)

Eliminating 't' using (i) and (ii), we get

$$R\equiv\left(5, -\frac{1}{2}\right)\therefore PR=\frac{3}{2}\sqrt{5}$$

16 (b) :



Let the rectangle be $ABCD$.

$A \equiv (t, 0), B \equiv (t, e^{-t}), C \equiv (-t, e^{-t}), D = (-t, 0)$

Area of $ABCD = 2te^{-t} = f(t)$ (say)

Now, $\frac{df}{dt}=2(t(-e^{-t})+e^{-t})=2e^{-t}(1-t)$

$$\frac{df}{dt}>0\Rightarrow t\in(0,1), \frac{df}{dt}<0\Rightarrow t\in(1,\infty)$$

$\Rightarrow$ $t = 1$ is point of maxima.

$\therefore$ Maximum area $= f(1)=\frac{2}{e}$.

17. (c)

18. (c) : We have, $\frac{dy}{dx}=y\tan x-y^2\sec x$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$$

Putting $\frac{1}{y}=v \Rightarrow \frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$, we obtain

$\frac{dv}{dx}+\tan x\cdot v=\sec x$, which is linear D.E.

I.F. $= e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

∴ The solution is given by

$$v \sec x = \int \sec^2 x dx + c \Rightarrow \frac{1}{y}\sec x = \tan x + c$$

$\Rightarrow$ $\sec x = y(c + \tan x)$

19. (c) : $A(z) = A\left(\frac{x+y}{1+xy}\right)$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]\begin{bmatrix} 1 & -\left(\frac{x+y}{1+xy}\right) \\ -\left(\frac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

∴ $A(x) \cdot A(y) = A(z)$

20. (c) : We have, $AB^2 = BA \Rightarrow A^5(AB^2) = A^5BA$

$\Rightarrow B^2 = A^5BA$

$\Rightarrow B^4 = (A^5BA)(A^5BA) = A^5B^2A = A^5(A^5BA)A$

$\Rightarrow B^4 = A^4BA^2$

$\Rightarrow B^8 = (A^4BA^2)(A^4BA^2) = A^4B^2A^2 = A^4(A^5BA)A^2$

$\Rightarrow B^8 = A^3BA^3$

$\Rightarrow B^{16} = (A^3BA^3)(A^3BA^3) = A^3B^2A^3 = A^3(A^5BA)A^3$
$= A^2BA^4$

$\Rightarrow B^{32} = (A^2BA^4)(A^2BA^4) = A^2B^2A^4 = A^2(A^5BA)A^4$
$= ABA^5$

$\Rightarrow B^{64} = (ABA^5)(ABA^5) = AB^2A^5 = A(A^5BA)A^5 = B$

$\Rightarrow B^{63} = I$

21. (0.2) : $A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 5\alpha + 5\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 25\alpha + 5\alpha^2 \\ 0 & 0 & 25 \end{bmatrix}$$

Given, $|A^2| = 25 \Rightarrow 625\alpha^2 = 25 \Rightarrow |\alpha| = \frac{1}{5} = 0.2$

22. (6) : Given $f(x+y) = f(x)f(y)$

∴ $f(0+0) = (f(0))^2$

$\Rightarrow f(0) = 0$ or $f(0) = 1$ but $f(0) \neq 0$

Now $f'(x) = \underset{h\to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{\text{Lt}} \frac{f(x)f(h) - f(x)}{h} = f(x) \underset{h\to 0}{\text{Lt}} \frac{f(h)-1}{h}$$

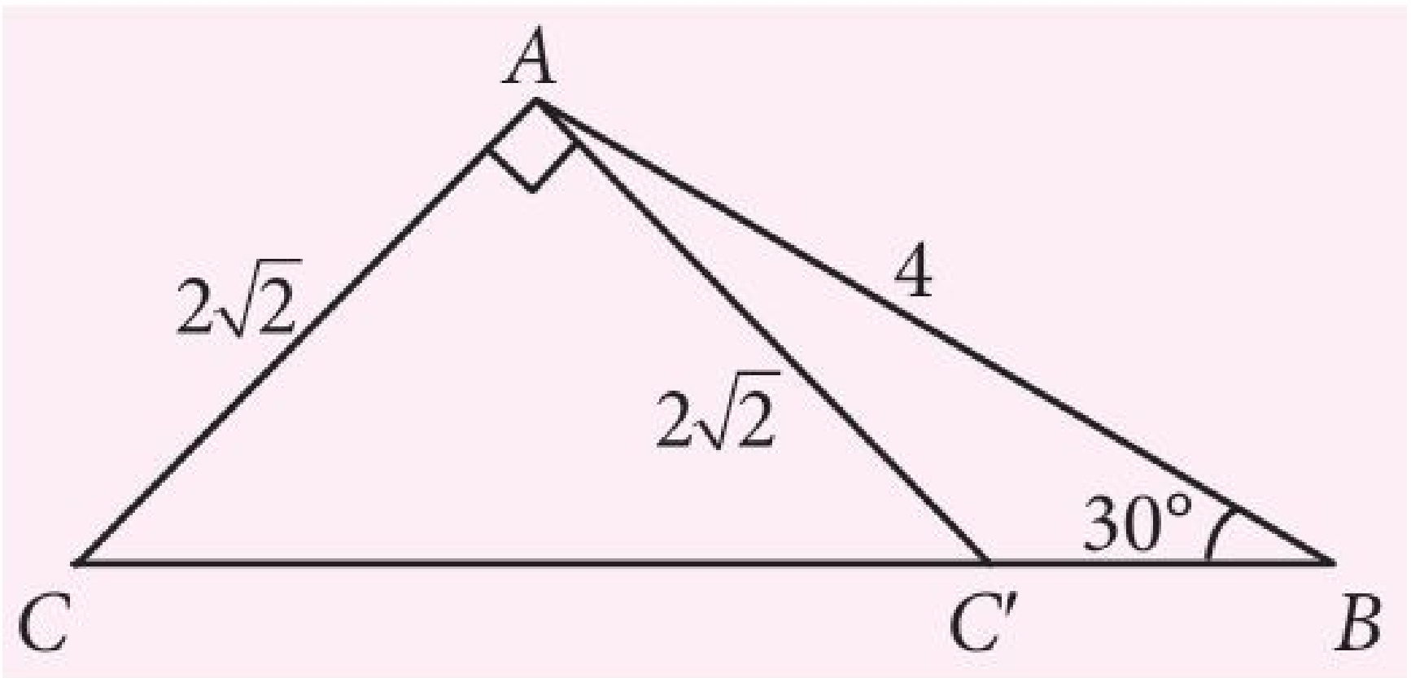
∴ $f'(0) = f(0) \underset{h\to 0}{\text{Lt}} \frac{f(h)-1}{h}$

$\Rightarrow 3 = \underset{h\to 0}{\text{Lt}} \frac{f(h)-1}{h}$ $\quad (\because f(0) = 1)$

Now, $f'(x) = f(x) \underset{h\to 0}{\text{Lt}} \frac{f(h)-1}{h}$

∴ $f'(5) = f(5) \times 3 = 2 \times 3 = 6$

23. (4) :



Let the third side be a. From cosine rule, we have

$(2\sqrt{2})^2 = 4^2 + a^2 - 2 \times 4a\cos 30°$

$$\Rightarrow 8 = 16 + a^2 - 8a\frac{\sqrt{3}}{2} \Rightarrow a^2 - 4a\sqrt{3} + 8 = 0$$

Giving $a = \frac{4\sqrt{3} \pm \sqrt{48 - 32}}{2} = \frac{4\sqrt{3} \pm 4}{2} = 2(\sqrt{3} \pm 1)$

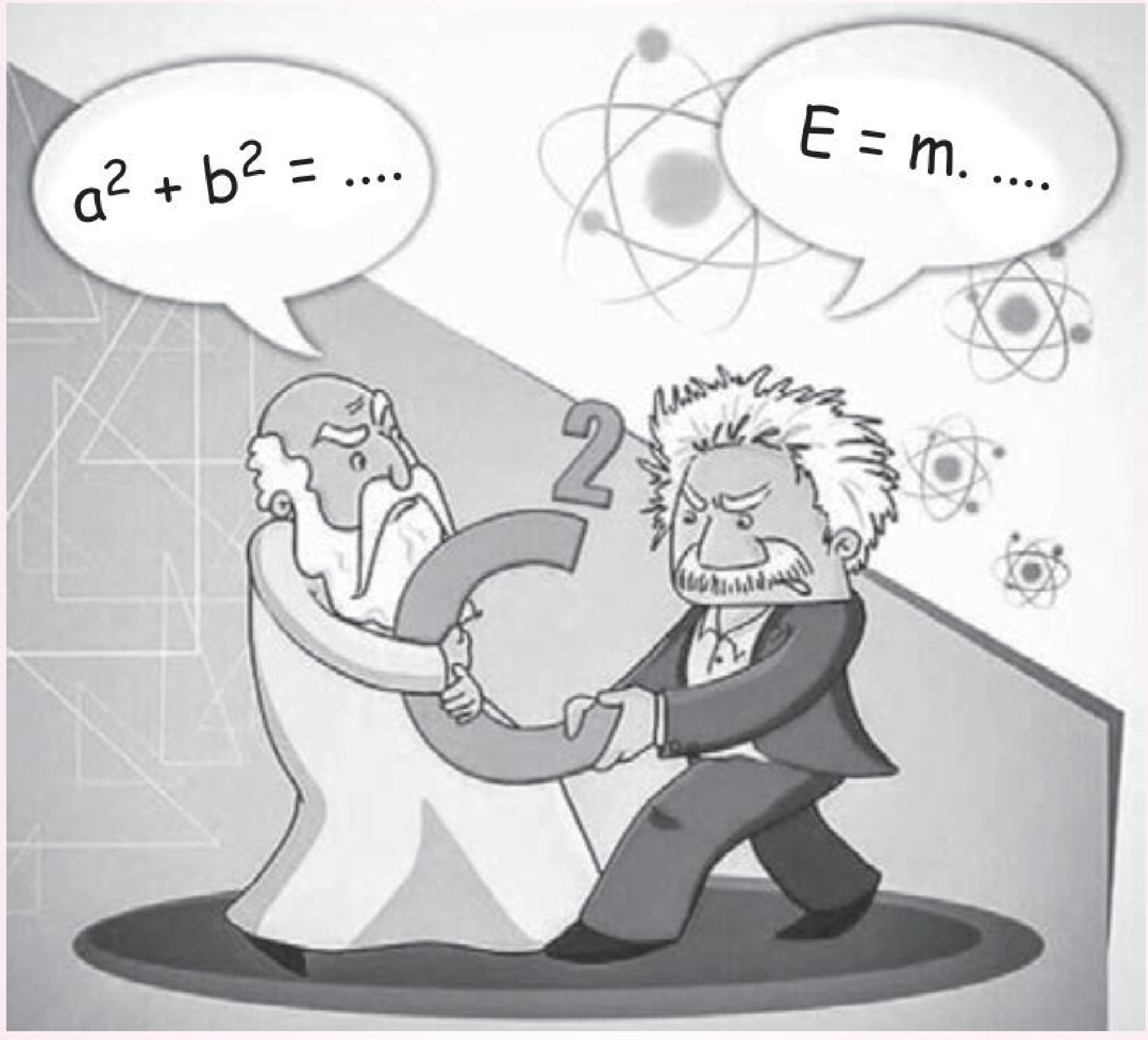
Thus, $BC = 2(\sqrt{3}+1),\ BC' = 2(\sqrt{3}-1)$

Difference of areas

$$= \frac{1}{2}\cdot 2(\sqrt{3}+1)\cdot 4 \cdot \sin 30° - \frac{1}{2}\cdot 2(\sqrt{3}-1)\cdot 4\cdot \sin 30°$$

$$= \frac{1}{2}\cdot 4 \times \frac{1}{2} \times 2[2] = 4$$

COMIC CAPSULE



24. (2) : $y = -2x + 1$ passes through nearest directrix

$$\Rightarrow 0=\frac{-2a}{e}+1\Rightarrow \frac{2a}{e}=1\Rightarrow \frac{a}{e}=\frac{1}{2}\Rightarrow e=2a$$

Now, $e^2=1+\frac{b^2}{a^2}\Rightarrow e^2=5-\frac{4}{e^2}\Rightarrow e^4-5e^2+4=0$

$\Rightarrow (e^2-1)(e^2-4)=0 \quad \therefore\ e^2=4 \Rightarrow e=2$

25. (0.8) : From the given table prime numbers are 2, 3, 5 and 7.

$\therefore\ P(E) = P(2 \text{ or } 3 \text{ or } 5 \text{ or } 7)$

$= P(2) + P(3) + P(5) + P(7) = 0.62$

Now, $P(F) = P(1 \text{ or } 2 \text{ or } 3)$

$= P(1) + P(2) + P(3) = 0.50$

$\therefore\ P(E \cap F) = P(2 \text{ or } 3) = P(2) + P(3) = 0.35$

Now, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.62 + 0.50 - 0.35 = 0.8$

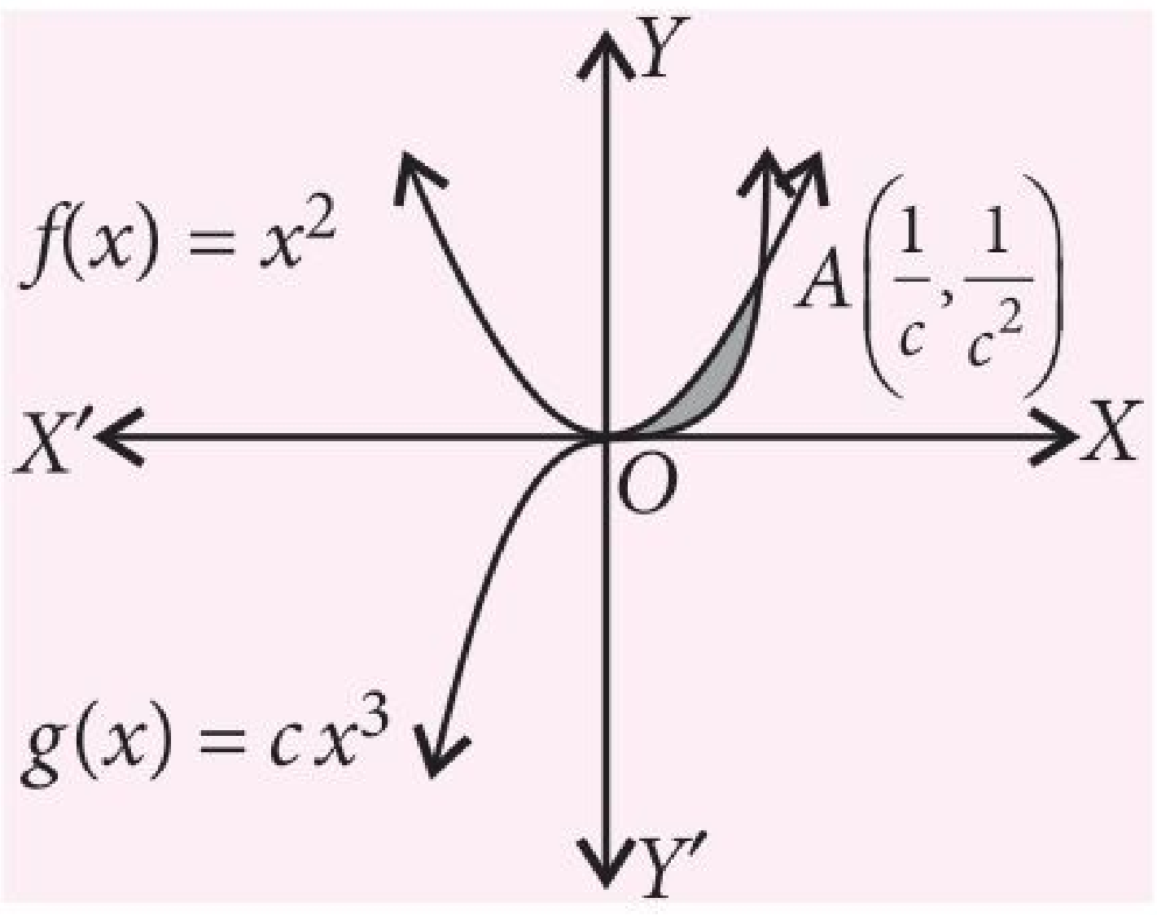
26. (7) : Given that $I_n=\int_0^{\pi/4}\tan^n x\,dx$

$$\therefore\ I_8+I_6=\int_0^{\pi/4}(\tan^8 x+\tan^6 x)dx$$

$$=\int_0^{\pi/4}\tan^6 x(\tan^2 x+1)\,dx=\int_0^{\pi/4}\tan^6 x\,d(\tan x)$$

$$=\left[\frac{\tan^7 x}{7}\right]_0^{\pi/4}=\frac{1}{7}$$

27. (6) : The two curves $f(x) = x^2$ and $g(x) = cx^3$ intersect at $A\left(\frac{1}{c},\frac{1}{c^2}\right)$ and $O\,(0, 0)$.



Given, $\int_0^{1/c}(x^2-cx^3)dx=\frac{2}{3}$

$$\Rightarrow \left[\frac{x^3}{3}-\frac{cx^4}{4}\right]_0^{\frac{1}{c}}=\frac{2}{3}\Rightarrow \frac{1}{12c^3}=\frac{2}{3}\Rightarrow c=\frac{1}{2}$$

Hence $\frac{1}{c}+\frac{1}{c^2}=2+4=6$

28. (3) : The given differential equation is

$$\frac{dy}{dx}=\frac{xy}{x^2+y^2},\ y(1)=1$$

Putting $y=vx\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow\ v+x\frac{dv}{dx}=\frac{v}{1+v^2}\ \Rightarrow\ x\frac{dv}{dx}=\frac{v}{1+v^2}-v=\frac{-v^3}{1+v^2}$$

$$\Rightarrow\ \int\frac{1+v^2}{v^3}dv=-\int\frac{dx}{x}\Rightarrow -\frac{1}{2v^2}+\log|v|=-\log|x|+C$$

$$\Rightarrow\ -\frac{x^2}{2y^2}+\log\left|\frac{y}{x}\right|+\log|x|=C \qquad \text{...(i)}$$

Putting $x = 1$, $y = 1$ in (i), we get $C = -1/2$

Now, (i) becomes $\log y-\frac{x^2}{2y^2}=-\frac{1}{2}$ ...(ii)

Putting $x = x_0$, $y = e$ in (ii), we get $x_0=\sqrt{3}e$

29. (6) : We have, $6\int_1^x f(t)\,dt=3xf(x)-x^3$

Differentiating both sides with respect to x, we get

$$6f(x)=3f(x)+3xf'(x)-3x^2$$

$\Rightarrow\ 3f(x) = 3xf'(x) - 3x^2$

$$\Rightarrow\ f'(x)-\frac{1}{x}f(x)=x\ \Rightarrow\ \frac{d}{dx}\{f(x)\}+\left(-\frac{1}{x}\right)f(x)=x$$

Here I.F. $=e^{\int-\frac{1}{x}dx}=e^{-\log x}=1/x$

$\therefore$ The general solution is $\frac{f(x)}{x}=\int x\times\frac{1}{x}dx=x+C$

$\Rightarrow\ f(x) = x^2 + Cx$

$\therefore\ f(1) = 1 + C \Rightarrow 2 = 1 + C \Rightarrow C = 1$

Thus, $f(x) = x^2 + x \quad \therefore\ f(2) = 4 + 2 = 6$

30. (8) : Let the coin be tossed n times and X be the random variable which denotes number of heads obtained in n tosses of the coin. Then

$$P(X=r)={}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r}={}^nC_r\left(\frac{1}{2}\right)^n;\ r=0,1,2,\ldots,n$$

Given that $P(X\ge 2)\ge 0.96\Rightarrow\{1-P(X<2)\}\ge 0.96$

$$\Rightarrow\ \{1-P(X=0)-P(X=1)\}\ge 0.96$$

$$\Rightarrow\ \left(1-{}^nC_0\left(\frac{1}{2}\right)^n-{}^nC_1\left(\frac{1}{2}\right)^n\right)\ge 0.96$$

$$\Rightarrow\ \frac{1}{2^n}(2^n-{}^nC_0-{}^nC_1)\ge 0.96\Rightarrow 1-\frac{n+1}{2^n}\ge 0.96$$

$\Rightarrow\ (n + 1) \le (0.04)2^n \ \Rightarrow\ n = 8, 9, 10, \ldots$

Hence, the least value of n is 8.

◆◆

PRACTICE PAPER

BITSAT

1. The distance between the lines $\vec{r}=(4\hat{i}-7\hat{j}-9\hat{k})+t(3\hat{i}-7\hat{j}+4\hat{k})$ and $\vec{r}=(7\hat{i}-14\hat{j}-5\hat{k})+s(-3\hat{i}+7\hat{j}-4\hat{k})$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) 0

2. In a triangle ABC, if $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then $A=$

(a) 30° (b) 45° (c) 60° (d) 90°

3. If α, β, γ are roots of $x^3+4x+1=0$, then $(\alpha+\beta)^{-1}+(\beta+\gamma)^{-1}+(\gamma+\alpha)^{-1}$ equals

(a) 2 (b) 3 (c) 4 (d) 5

4. If $f(x)=A\sin\left(\frac{\pi x}{2}\right)+B, f'\left(\frac{1}{2}\right)=\sqrt{2}$ and $\int_0^1 f(x)dx=\frac{2A}{\pi}$, then the constant A and B are, respectively

(a) $\pi/2$ and $\pi/2$ (b) $2/\pi$ and $3/\pi$

(c) 0 and $-4/\pi$ (d) $4/\pi$ and 0

5. If at any point (x, y) on a curve, subtangent and subnormal are of equal length, then the length of tangent is

(a) $\sqrt{2y}$ (b) $|\sqrt{2}y|$

(c) $|y|$ (d) none of these

6. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then value of $4x^2-4xy\cos\alpha+y^2$ is equal to

(a) 4 (b) $2\sin 2\alpha$

(c) $-4\sin^2\alpha$ (d) $4\sin^2\alpha$

7. If $X_1=(x_1, y_1)$ and $X_2=(x_2, y_2)$ are two optimal solutions of an L.P.P., then

(a) $\lambda X_1+(1-\lambda)X_2, \lambda\in R$, is also an optimal solution

(b) $\lambda X_1+(1+\lambda)X_2, \lambda\in R$, is also an optimal solution

(c) $\lambda X_1+(1+\lambda)X_2, 0\le\lambda\le 1$, is also an optimal solution

(d) $\lambda X_1+(1-\lambda)X_2, 0\le\lambda\le 1$, is also an optimal solution

8. The function $f(x)=\frac{\tan([x-\pi]\pi)}{1+[x]^2}$ is

(a) discontinuous at some x

(b) continuous at all x, but $f'(x)$ does not exist for some x

(c) $f'(x)$ exist for all x, but $f''(x)$ does not exist for some x

(d) $f''(x)$ exists for all x.

9. There are two groups of subjects, group I consists of 5 science subjects and 3 engineering subjects while group II consists of 3 science and 5 engineering subjects. An unbiased die is cast. If number 3 or 5 turns up, a subject from group I is selected, otherwise a subject is selected from group II. The probability that an engineering subject is selected ultimately is

(a) $\frac{7}{13}$ (b) $\frac{9}{17}$

(c) $\frac{13}{24}$ (d) $\frac{11}{20}$

10. For any two real numbers a and b, we define aRb if and only if $\sin^2 a+\cos^2 b=1$. The relation R is

(a) reflexive but not symmetric

(b) symmetric but not transitive

(c) transitive but not reflexive

(d) an equivalence relation

11. If $5\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+7\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\tan^{-1}\left(\frac{2x}{1-x^2}\right)-\tan^{-1}x=5\pi,$ then x is equal to

(a) 3 (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

12. The locus of the centre of the circle for which one end of diameter is (3, 3) while the other end lies on the line $x + y = 4$ is $x + y = k$, then k equals

(a) 9 (b) 5 (c) 3 (d) 2

13. There are 3 copies each of 4 different books. The number of ways they can be arranged in a shelf is

(a) 369600(b) 400400 (c) 420600 (d) 440720

14. If α and β ($\alpha < \beta$), are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$
(b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$
(d) $\alpha < 0 < |\alpha| < \beta$

15. Let z and ω be two complex numbers such that $|z| \le 1$, $|\omega| \le 1$ and $|z - i\bar{\omega}| = |z + i\omega| = 2$, then z equals

(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

16. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation

(a) $0 < M \le 1$ (b) $1 \le M \le 2$
(c) $2 \le M \le 3$ (d) $3 \le M \le 4$

17. If C_r stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\left[C_0^2-2C_1^2+\ldots\ldots\ldots+(-1)^n(n+1)C_n^2\right],$$

where n is an even positive integer, is equal to

(a) 0 (b) $(-1)^{\frac{n}{2}}(n+1)$
(c) $(-1)^n(n + 2)$ (d) none of these

18. In arrangements of the word 'MONDAY', if p = number of words that do not begin with M and q = number of words which begin with M but do not end with Y, then $p : q$ =

(a) 25 : 4 (b) 4 : 25
(c) < 0 (d) none of these

19. If $S_n=\sum_{r=1}^{n}2^{r-1}$, then $\sum_{r=1}^{n}\log_{2^r}(S_r+1)=$

(a) n (b) $\frac{n}{2}$
(c) $\frac{n(n+1)}{2}$ (d) none of these

20. If $A=\begin{bmatrix}\sqrt[3]{8} & 0 & 0\\ 0 & \sqrt[3]{2} & 0\\ 0 & 0 & \frac{1}{\sqrt[3]{4}}\end{bmatrix}$ and $|\text{adj}(\text{adj}A)|=\sqrt[3]{\lambda^\lambda}$, then λ =

(a) 12 (b) 4
(c) 2 (d) none of these

21. If $A=\begin{pmatrix}0 & -\tan\left(\frac{\theta}{2}\right)\\ \tan\left(\frac{\theta}{2}\right) & 0\end{pmatrix}$ and I be unit matrix of order 2, then $(I-A)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=$

(a) I (b) $I + A$ (c) $A - I$ (d) 0

22. If n_1 and n_2 be randomly chosen from first 50 positive integers and probability that $3^{n_1}+3^{n_2}$ and $8^{n_1}+8^{n_2}$ each is divisible by 5 be p_1 and p_2 respectively, then

(a) $p_1 = p_2$
(b) $p_1 < p_2$
(c) $p_1 > p_2$
(d) Nothing can be said

23. If $\frac{a^2+1-\cos^2 x}{\cos 2x}+\frac{a^2}{1-\cot^2 x}=0$ has atleast one real solution of x, then a lies in

(a) $(-1, 1)$ (b) $[-1, 1]$
(c) $\{0\}$ (d) none of these

24. If $A=\begin{bmatrix}6 & 8 & 5\\ 4 & 2 & 3\\ 9 & 7 & 1\end{bmatrix}$ is the sum of a symmetric matrix B and skew-symmetric matrix C, then B is

(a) $\begin{bmatrix}6 & 6 & 7\\ 6 & 2 & 5\\ 7 & 5 & 1\end{bmatrix}$ (b) $\begin{bmatrix}0 & 2 & -2\\ -2 & 5 & -2\\ 2 & 2 & 0\end{bmatrix}$

(c) $\begin{bmatrix}6 & 6 & 7\\ -6 & 2 & -5\\ -7 & 5 & 1\end{bmatrix}$ (d) $\begin{bmatrix}0 & 6 & -2\\ 2 & 0 & -2\\ -2 & -2 & 0\end{bmatrix}$

25. The system $y^{(x^2+7x+12)} = 1$ and $x + y = 6$, $y > 0$ has
(a) no solution
(b) one solution
(c) two solutions
(d) More than two solutions

26. The differential equation of the family of parabolas with focus at the origin and the x-axis as axis, is

(a) $y\left(\frac{dy}{dx}\right)^2 + 4x\frac{dy}{dx} = 4y$

(b) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$

(c) $y\left(\frac{dy}{dx}\right)^2 + y = 2x\frac{dy}{dx}$

(d) $y\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y = 0$

27. Equation of the line passing through the point of intersection of lines $2x - 3y + 4 = 0$, $3x + 4y - 5 = 0$ and perpendicular to line $6x - 7y + 3 = 0$, is
(a) $119x + 102y + 125 = 0$
(b) $119x + 102y = 125$
(c) $119x - 102y = 125$
(d) none of these

28. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$ is equal to
(a) 0
(b) 1
(c) 6
(d) 12

29. If $x_1, x_2, \ldots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \ldots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle number, is

(a) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$
(b) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

(c) $\frac{{}^{19}C_2 \times {}^{31}C_2}{{}^{50}C_5}$
(d) none of these

30. Matrix A is such that $A^2 = 2A - I$, where I is the identity matrix. Then, for $n \geq 2$, A^n is equal to
(a) $nA - (n - 1)I$
(b) $nA - I$
(c) $2^{n-1}A - (n - 1)I$
(d) $2^{n-1}A - I$

31. If C_r stands for ${}^nC_r$ and $\frac{C_0}{1} - \frac{C_1}{3} + \frac{C_2}{5} - \ldots$ $\ldots + \frac{(-1)^n C_n}{2n+1} = k\int_0^1 x(1 - x^2)^{n-1}dx$, then k is equal to

(a) $\frac{2^{2n+1} \cdot n \cdot n!}{(2n+1)!}$
(b) $2^{2n} \cdot \frac{n!}{(2n+1)!}$

(c) ${}^{2n+1}C_n$
(d) $\frac{2^{2n+1} \cdot n \cdot (n!)^2}{(2n+1)!}$

32. The equation $3\sin^2 x + 10\cos x - 6 = 0$ is satisfied, if

(a) $x = n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$
(b) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

(c) $x = n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$
(d) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$

33. The number of all possible triplets (x, y, z) such that $(x + y) + (y + 2z)\cos 2\theta + (z - x)\sin^2\theta = 0$ for all θ is
(a) 0
(b) 1
(c) 3
(d) infinite

34. Solution set of the inequality $\sqrt{x} - 3 \leq \frac{2}{\sqrt{x} - 2}$ is
(a) $2 \leq x \leq 3$
(b) $[0, 1] \cup [2, 16]$
(c) $[0, 1] \cup [3, 16]$
(d) none of these

35. The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$, are
(a) $(2, -4)$
(b) $(18, -12)$
(c) $(2, 4)$
(d) none of these

36. Let $\vec{b} = 4\hat{i} + 3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$ respectively, are given by

(a) $2\hat{i} - \hat{j}, -\frac{2}{5}\hat{i} + \frac{11}{5}\hat{j}$
(b) $2\hat{i} + \hat{j}, \hat{i} + \hat{j}$

(c) $\hat{i} - \hat{j}, \hat{i} + \hat{j} + \hat{k}$
(d) $\hat{i} + \hat{j} + \hat{k}, \hat{i} - \hat{k}$

37. Given the points $A(0, a)$ and $B(0, -a)$, the equation of the locus of point $P(x, y)$ such that $|AP - BP| = 6$ is

(a) $\frac{x^2}{a^2 - 9} - \frac{y^2}{9} = 1$
(b) $\frac{x^2}{a^2 - 9} - \frac{y^2}{9} + 1 = 0$

(c) $\frac{x^2}{9} - \frac{y^2}{a^2 - 9} = 1$
(d) $\frac{x^2}{9} - \frac{y^2}{a^2 - 9} + 1 = 0$

38. If $z_r = \cos\frac{r\alpha}{n^2} + i\sin\frac{r\alpha}{n^2}$, where $r = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} z_1 z_2 z_3 \ldots z_n$ is equal to

(a) $\cos\alpha + i\sin\alpha$
(b) $\cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right)$

(c) $e^{i\alpha/2}$
(d) $\sqrt[3]{e^{i\alpha}}$

39. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is

(a) $x^2 + y^2 - 2x + 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 + 2x - 2y - 23 = 0$

40. The arbitrary constant on which the value of

$$\Delta = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \cos(p-d)a & \cos pa & \cos(p-d)a \\ \sin(p-d)a & \sin pa & \sin(p-d)a \end{vmatrix}$$

does not depend, is

(a) α (b) p (c) d (d) a

41. Domain of $f(x) = \sin^{-1}[2 - 4x^2]$([·] denotes the greatest integer function) is

(a) $[-1, 1]$ (b) $(-2, 2)$
(c) $\left[-\frac{\sqrt{3}}{2}, 0\right) \cup \left(0, \frac{\sqrt{3}}{2}\right]$ (d) none of these

42. The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants is

(a) 5 (b) 6 (c) 3 (d) 2

43. The angle between the planes whose vector equations are $\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i}-3\hat{j}+5\hat{k}) = 3$ is

(a) $\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$ (b) $\cos^{-1}\left(\frac{15}{\sqrt{7}}\right)$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

44. In a random experiment, the success is thrice that of failure. If the experiment is repeated 5 times, then the probability that atleast 4 times favourable is

(a) $\frac{1296}{2048}$ (b) $\frac{1003}{2048}$
(c) $\frac{1203}{2048}$ (d) none of these

45. Evaluate : $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

(a) $-\frac{1}{3}\log|1+\tan\theta| - \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$

(b) $-\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$

(c) $-\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta + \tan\theta + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta + 1}{\sqrt{3}}\right) + C$

(d) none of these

SOLUTIONS

1. (d) : Given lines are

$$L_1 \equiv \vec{r} = (4\hat{i} - 7\hat{j} - 9\hat{k}) + t(3\hat{i} - 7\hat{j} + 4\hat{k})$$

$$L_2 \equiv \vec{r} = (7\hat{i} - 14\hat{j} - 5\hat{k}) - s(3\hat{i} - 7\hat{j} + 4\hat{k})$$

Comparing these with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\,\vec{b}_2$, we get $\vec{b} = \vec{b}_1 = \vec{b}_2 = 3\hat{i} - 7\hat{j} + 4\hat{k}$

So both lines are parallel.

$\therefore$ Distance between the lines $= \left|\frac{(\vec{a}_2 - \vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$

$$= \left|\frac{(3\hat{i} - 7\hat{j} + 4\hat{k})\times(3\hat{i} - 7\hat{j} + 4\hat{k})}{|3\vec{i} - 7\vec{j} + 4\vec{k}|}\right| = 0$$

2. (d) : By using cosine rule

$$\frac{2(b^2+c^2-a^2)+(c^2+a^2-b^2)+2(a^2+b^2-c^2)}{2abc} = \frac{a^2+b^2}{abc}$$

$\Rightarrow\ 3b^2 + c^2 + a^2 = 2a^2 + 2b^2$

$\Rightarrow\ a^2 = b^2 + c^2 \ \Rightarrow\ A = 90°$

3. (c): If α, β, γ are roots of $x^3 + ax^2 + bx + c = 0$, then

$$(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = \frac{a^2+b}{-ab+c} = \frac{a^2+b}{c-ab}$$

Now in the given problem, we have $a = 0$, $b = 4$, $c = 1$

$$\Rightarrow\ \frac{a^2+b}{c-ab} = \frac{4}{1}$$

4. (d) : $f'(x) = A\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$

$$\Rightarrow\ \sqrt{2} = f'(1/2) = A\frac{\pi}{2}\left(\frac{1}{\sqrt{2}}\right) \ \Rightarrow\ A = \frac{4}{\pi}$$

Now, $\int_0^1 f(x)dx = \frac{2A}{\pi} \quad \Rightarrow \quad \frac{2A}{\pi} = \frac{4}{\pi}\int_0^1 \sin\left(\frac{\pi}{2}x\right)dx + \int_0^1 Bdx$

$$\Rightarrow \frac{2A}{\pi} = -\frac{4}{\pi}\left[\frac{\cos(\pi x/2)}{\pi/2}\right]_0^1 + B \Rightarrow \frac{2A}{\pi} = \frac{4}{\pi}\times\frac{2}{\pi} + B \Rightarrow B = 0$$

5. (b) : Given, $yy' = \frac{y}{y'} \Rightarrow y'^2 = 1 \Rightarrow y' = \pm 1$

$\therefore$ Length of tangent is $\frac{y}{y'}\sqrt{1+(y')^2} = |\sqrt{2}y|$

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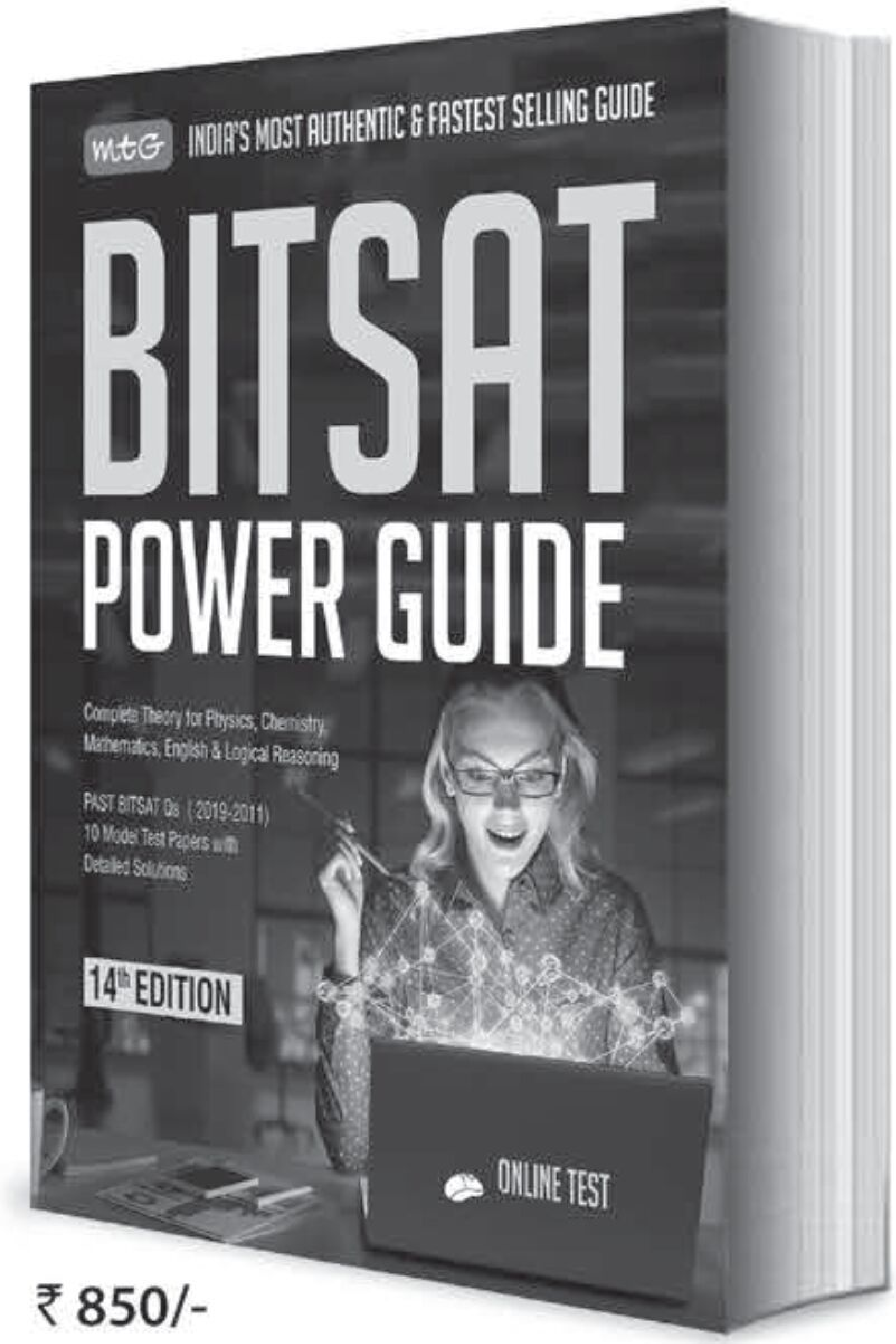
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6. (d) : Given $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$$\Rightarrow \frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha$$

$$\Rightarrow \left(\cos\alpha - \frac{xy}{2}\right)^2 = (1-x^2)\left(1-\frac{y^2}{4}\right)$$

$$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4(1-\cos^2\alpha) = 4\sin^2\alpha.$$

7. (d) : If X_1 and X_2 are optimal solutions of an L.P.P., then $\lambda X_1 + (1-\lambda)X_2$, $0 \le \lambda \le 1$ is also an optimal solution.

8. (d) : $\tan n\pi = 0\ \forall\ n \in Z\ \therefore\ f(x) = 0\ \forall x$, a constant function. $\therefore\ f''(x) = 0\ \forall x$.

9. (c): Let E be the event that engineering subject is selected and A, B be the events that group I and II is selected respectively.

Now, $P(A) = 2/6$, $P(B) = 4/6$

$$\therefore\quad P(E) = P(A) \times P(E/A) + P(B) \times P(E/B)$$

$$= \frac{2}{6}\cdot\frac{3}{8} + \frac{4}{6}\cdot\frac{5}{8} = \frac{13}{24}$$

10. (d) : $\because\ aRb$ if $\sin^2 a + \cos^2 b = 1$

$\because\quad \sin^2 a + \cos^2 a = 1$ for all $a \in R$

$\therefore\quad aRa \Rightarrow R$ is reflexive

$\because\quad \sin^2 a + \cos^2 b = 1 \Rightarrow 1 - \cos^2 a + 1 - \sin^2 b = 1$

$\Rightarrow\quad \sin^2 b + \cos^2 a = 1 \Rightarrow bRa$

$\therefore\quad R$ is symmetric

Further, let aRb and bRc

i.e. $\sin^2 a + \cos^2 b = 1$ … (i)

and $\sin^2 b + \cos^2 c = 1$ … (ii)

Adding, $\sin^2 a + 1 + \cos^2 c = 2$

$\Rightarrow\quad \sin^2 a + \cos^2 c = 1 \Rightarrow aRc$

$\therefore\quad R$ is also transitive.

$\therefore\quad R$ is an equivalence relation.

11. (d): Given,

$$5\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 7\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \tan^{-1}(x) = 5\pi$$

Putting $x = \tan\theta$, we get

$$5\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + 7\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) - 4\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) - \tan^{-1}(\tan\theta) = 5\pi$$

$$\Rightarrow\quad 5 \times 2\theta + 7 \times 2\theta - 4 \times 2\theta - \theta = 5\pi$$

$$\Rightarrow\quad 15\theta = 5\pi \quad\Rightarrow\quad \theta = \pi/3$$

$$\therefore\ x = \tan\theta = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

12. (b) : Let (x, y) be the locus of centre.

$$\therefore\quad (x, y) = \frac{(3,3) + (x_0, 4 - x_0)}{2},$$

where x_0, y_0 lies on $x + y = 4$.

$$\Rightarrow\quad x = \frac{3 + x_0}{2};\ y = \frac{3 + 4 - x_0}{2}$$

$\therefore\quad 2x = 3 + x_0;\ 2y = 7 - x_0$

Adding equations, we get

$2x + 2y = 10 \Rightarrow x + y = 5$ is the required locus.

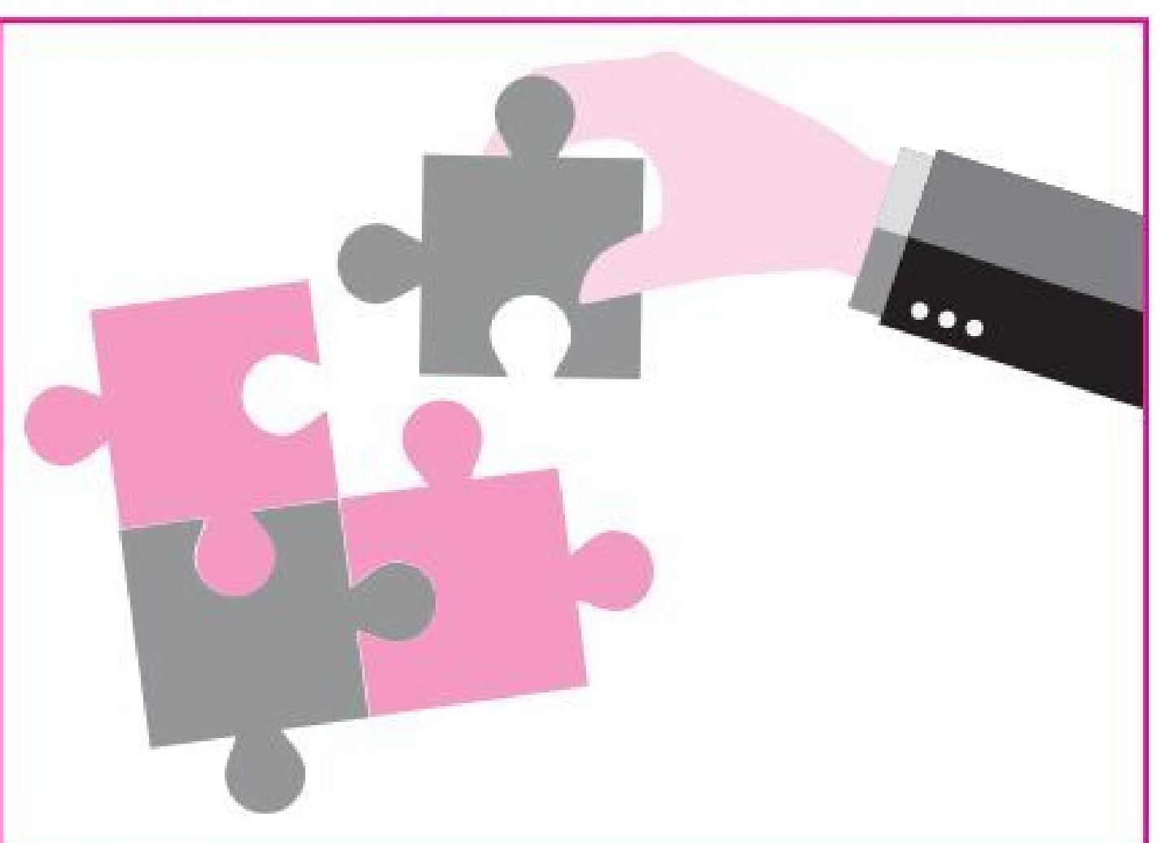
$\therefore\quad k = 5$

13. (a) : There are 4 sets of 3 identical books. The desired number is $\frac{12!}{(3!)^4} = 369600$

14. (b) **15. (c)** **16. (a)** **17. (d)**
18. (a) **19. (a)** **20. (b)** **21. (b)**
22. (a) **23. (c)** **24. (a)** **25. (d)**
26. (b) **27. (b)** **28. (c)** **29. (b)**
30. (a) **31. (d)** **32. (b)** **33. (d)**
34. (d) **35. (a)** **36. (a)** **37. (b)**
38. (c) **39. (a)** **40. (b)** **41. (c)**
42. (c) **43. (a)** **44. (a)** **45. (b)**

◆◆

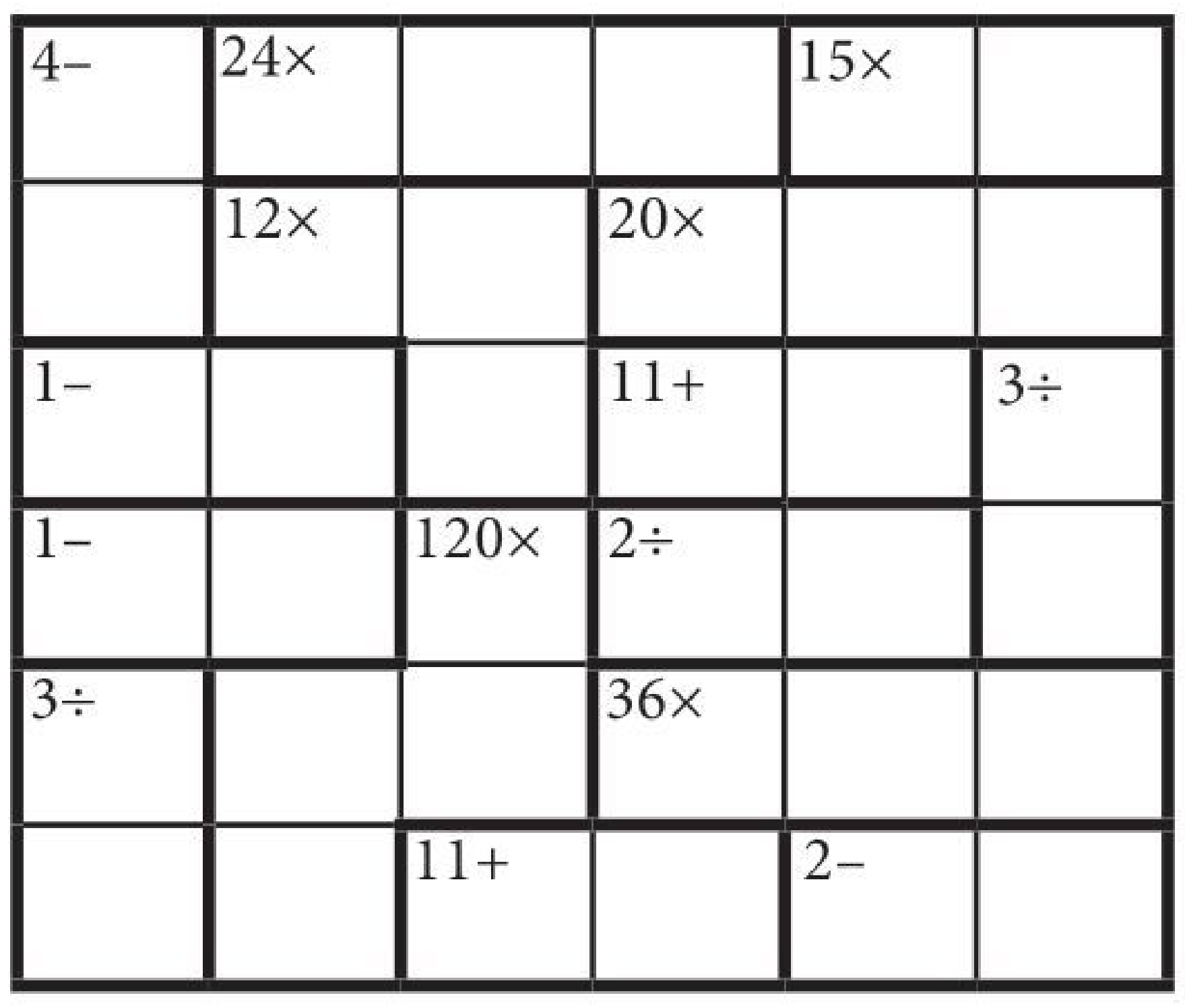
PUZZLE CORNER



MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics. In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.



4–	24×			15×	
	12×		20×		
1–			11+		3÷
1–		120×	2÷		
3÷			36×		
		11+		2–	

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OLYMPIAD CORNER

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1.$$
When does equality hold?

2. Suppose that u is a real parameter with $0 < u < 1$. Define
$$f(x)=\begin{cases}0 & \text{, if } 0\le x\le u\\ 1-\left(\sqrt{ux}+\sqrt{(1-u)(1-x)}\right)^2 & \text{, if } u\le x\le 1\end{cases}$$
and define the sequence $\{u_n\}$ recursively as follows: $u_1 = f(1)$, and $u_n = f(u_{n-1})$ for all $n > 1$. Show that there exists a positive integer k for which $u_k = 0$.

3. You have an unlimited supply of 5-gram and 8-gram weights that may be used in a pan balance. If you use only these weights and place them only in one pan, then find the largest number of grams that you cannot weigh.

4. Let A, B, C be vectors from the circumcenter of a triangle ABC to the respective vertices. Prove that
$$\frac{(B+C)|B-C|}{|B+C|}+\frac{(C+A)|C-A|}{|C+A|}+\frac{(A+B)|A-B|}{|A+B|}=0.$$

5. It is given that $\cos x = \cos y$ and $\sin x = -\sin y$. Prove that $\sin 1994x + \sin 1994y = 0$.

SOLUTIONS

1. We note first that
$$\frac{a^5+b^5}{2}\ge\left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right),$$

Since $a^5 - a^3b^2 - a^2b^3 + b^5$
$= (a - b)^2 (a + b) (a^2 + ab + b^2) \ge 0$ with equality if and only if $a = b$.

Similarly
$$\frac{a^3+b^3}{2}\ge\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)$$
because $a^3 - a^2b - ab^2 + b^3 = (a - b)^2 (a + b) \ge 0$, with equality if and only if $a = b$.

Thus $$\frac{a^5+b^5}{2}\ \ge\ \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right)\ge ab\left(\frac{a^3+b^3}{2}\right)$$
$$\ge\ ab\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)\ge\frac{a^2b^2(a+b)}{2}.$$

It is enough, therefore, to prove
$$\frac{ab}{ab(a+b)ab+ab}+\frac{bc}{bc(b+c)bc+bc}+\frac{ca}{ca(c+a)ca+ca}\le 1,$$
or $$\frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+\frac{1}{ca(c+a)+abc}\le 1.$$
Equivalently,
$$\frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ca(a+b+c)}\le 1,$$
or $$\frac{c}{abc(a+b+c)}+\frac{a}{abc(a+b+c)}+\frac{b}{abc(a+b+c)}\le 1.$$
Again, because $abc = 1$, we get
$$\frac{a+b+c}{a+b+c}\ \le\ 1,$$ which is true.

The equality requires $a = b = c = 1$.

2. Note first that $u_1 = 1 - u$. Since for all $x \in [u, 1]$, $u \le x$ and $1 - x \le 1 - u$, we have

$$1-\left(\sqrt{ux}+\sqrt{(1-u)(1-x)}\right)^2$$

$$=1-ux-(1-u)(1-x)-2\sqrt{ux(1-u)(1-x)}$$

$$=u+x-2ux-2\sqrt{ux(1-u)(1-x)}$$

$$\le u+x-2ux-2u(1-x)=x-u.$$

Therefore,

$$f(x)=0 \quad \text{if} \quad 0\le x\le u \qquad ...(1)$$

$$\text{and } f(x)\le x-u \quad \text{if} \quad u\le x\le 1. \qquad ...(2)$$

From (2) we get $u_2=f(u_1)-u=1-2u$ if $u_1\ge u$. An easy induction then yields $u_{n+1}=f(u_n)\le u_n-u\le 1-(n+1)u$ if $u_i\ge u$ for all i =1, 2, ...,n.

Thus for sufficiently large k, we must have $u_{k-1}<u$ and then $u_k=f(u_{k-1})=0$ by (1).

3. If W is the weight in grams then W can be weighed if W = 5x + 8y, where x and y are non-negative integers. Furthermore, y can be written in the form 5z + r, where z is a non-negative integer and r is either 0, 1, 2, 3 or 4. Thus W = 5(x + 8z) + 8(0), W = 5(x + 8z) + 8(1), W = 5(x + 8z) + 8(2), W = 5(x + 8z) + 8(3), or W = 5(x + 8z) + 8(5). These can be simplified to W = 5k, W = 5(k + 1) + 3, W = 5(k + 3) + 1, W = 5(k + 4) + 4 and W = 5(k + 6) + 2. By putting k = x + 8z, we conclude that W can be weighed if W = 5k + 8r, with $k \ge 0$ and $r \le 4$. Thus, according to the value of r, W = 5k, W = 5(k + 1) + 3, W = 5(k + 3) + 1, W = 5(k + 4) + 4, W = 5(k + 6) + 2. On the other hand, W (as well as every integer) can take exactly one of the forms: 5K, 5K + 1, 5K + 2, 5K + 3, or 5K + 4. By comparing these forms with the expressions for W that can be weighed, we conclude:

(i) if W is of the form 5K, then it can be weighed;

(ii) if W is of the form 5K + 1, then it can be weighed if K = k + 3; that is, when $K \ge 3$;

(iii) if W is of the form 5K + 2, then it can be weighed if K = k + 6; that is, when $K \ge 6$;

(iv) if W is of the form 5K + 3, then it can be weighed if K = k + 1; that is, when $K \ge 1$;

(v) if W is of the form 5K + 4, then it can be weighed if K = k + 4; that is, when $K \ge 4$.

Consequently, the largest values of W that cannot be weighed in categories: (ii), (iii), (iv) and (v), are 5(2) + 1 = 11, 5(5) + 2 = 27, 5(0) + 3 = 3, 5(3) + 4 = 19, respectively. The largest of them is 27.

4. We need to prove that

$$\frac{(B+C)|B-C|}{|B+C|}+\frac{(C+A)|C-A|}{|C+A|}+\frac{(A+B)|A-B|}{|A+B|}=0. \qquad ...(1)$$

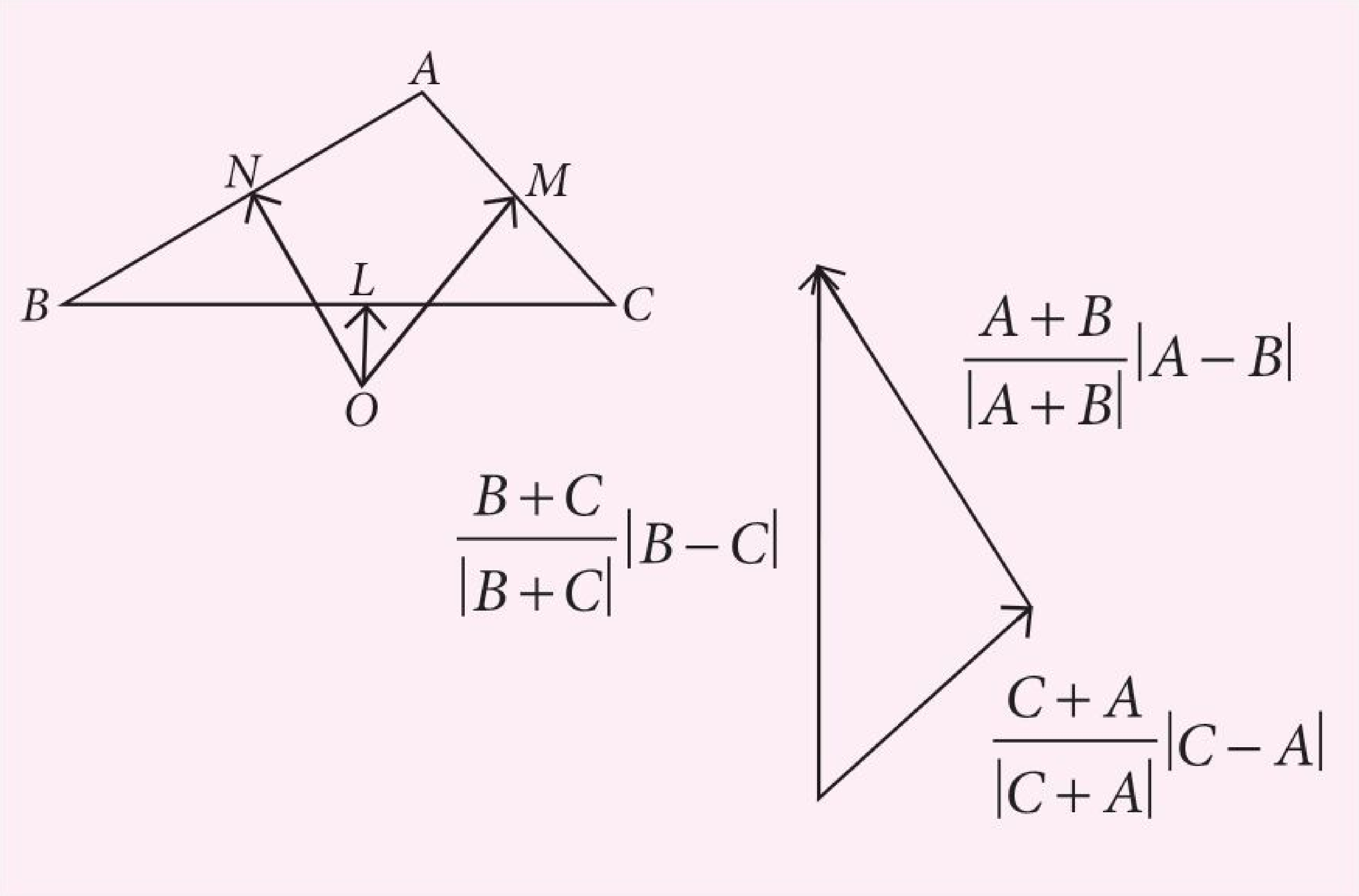
The lengths of

$$\frac{(B+C)}{|B+C|}|B-C|,\ \frac{(C+A)}{|C+A|}|C-A|,\ \frac{(A+B)}{|A+B|}|A-B|$$

are $|B-C|=a,\ |C-A|=b,\ |A-B|=c,$ so a triangle with sides equal to these lengths must be congruent to triangle ABC. With circumcentre as origin the directions of these vectors are perpendicular to BC, CA, AB respectively. So the given equation (1) is simply a mapping of the relation

$$(B-C)+(C-A)+(A-B)=0$$

under rotation by 90°. This is true always provided O is internal to triangle ABC. The figure below illustrates the situation for O an external point, where evidently a sign has to be adjusted appropriately.



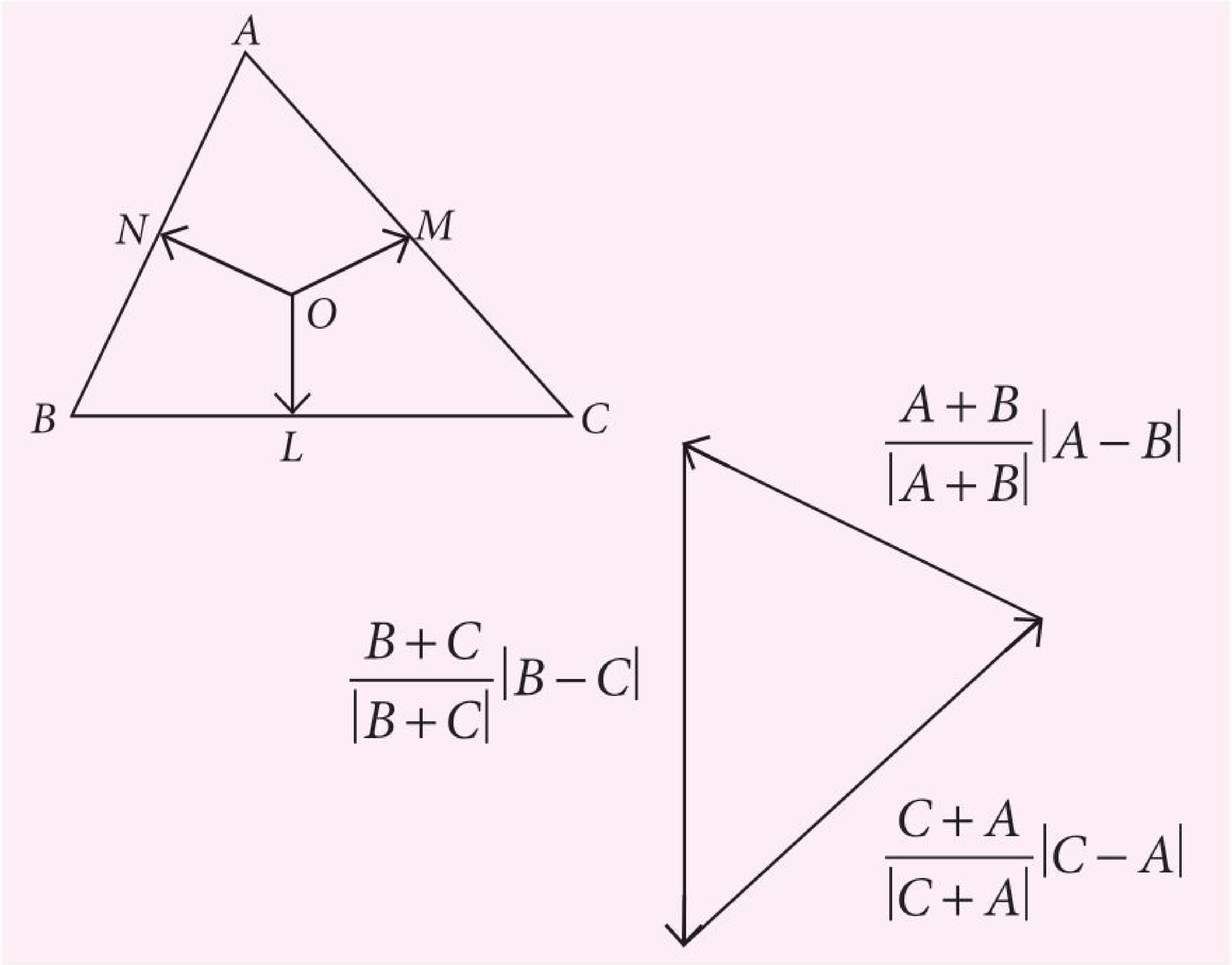
$\overrightarrow{OL}, \overrightarrow{OM}, \overrightarrow{ON}$ are in directions

$B + C$, $C + A$, $A + B$

It can be seen that for $\angle A$ obtuse one has

$$\frac{(C+A)|C-A|}{|C+A|}+\frac{(A+B)|A-B|}{|A+B|}=\frac{(B+C)|B-C|}{|B+C|}.$$

(When $\angle A$ = 90°, $B + C$ vanishes and the relation degenerates and requires further interpretation.) When O is internal the situation is satisfactory, as the diagrams below show.



Now relation (1) holds.

5. More generally, we prove that $\sin mx + \sin my = 0$, where m is an integer.

Since $\sin mx + \sin my = 2\sin\left(m\frac{x+y}{2}\right)\times$ $\cos\left(m\left(\frac{x-y}{2}\right)\right)$, it is sufficient to show that $\sin\left(\frac{x+y}{2}\right)=0$. Since $\sin\left(m\frac{x+y}{2}\right)=0$ follows easily by induction from

$$\sin\left((m+1)\frac{x+y}{2}\right)=\sin\left(m\frac{(x+y)}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$+\cos\left(m\frac{(x+y)}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

Now, $\cos x = \cos y \Leftrightarrow \cos x - \cos y = 0$

$$\Leftrightarrow -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}=0 \qquad \text{...(1)}$$

and $\sin x = -\sin y \Leftrightarrow \sin x + \sin y = 0$

$$\Leftrightarrow 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=0 \qquad \text{...(2)}$$

Squaring (1) and (2) and then adding, we find

$$4\sin^2\left(\frac{x+y}{2}\right)\underbrace{\left(\sin^2\frac{(x-y)}{2}+\cos^2\frac{(x-y)}{2}\right)}_{=1}=0$$

Hence $\sin\frac{x+y}{2}=0$.

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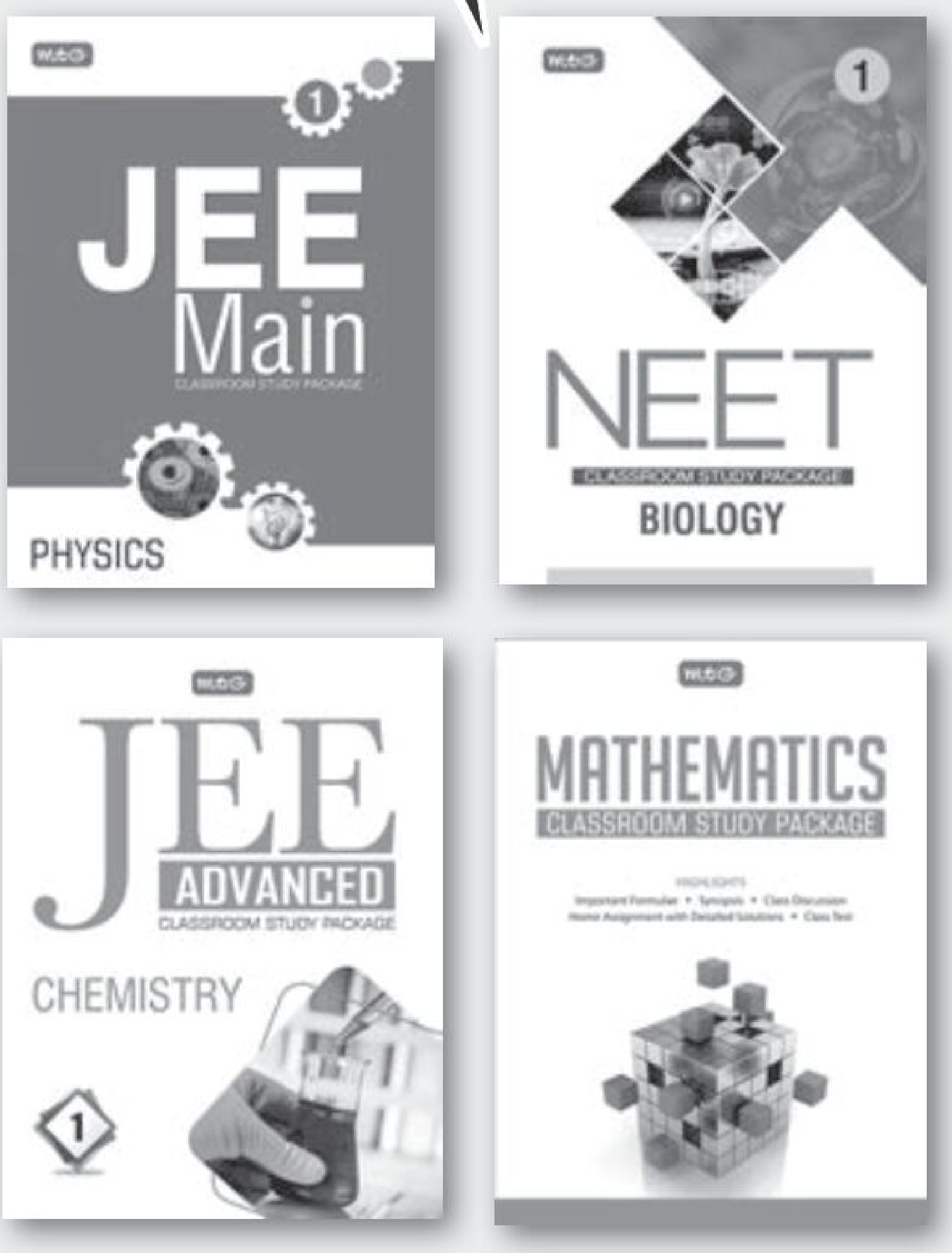

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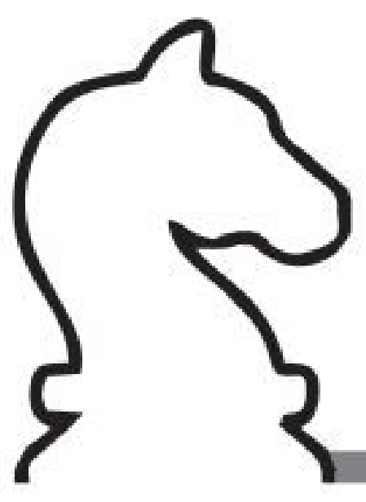
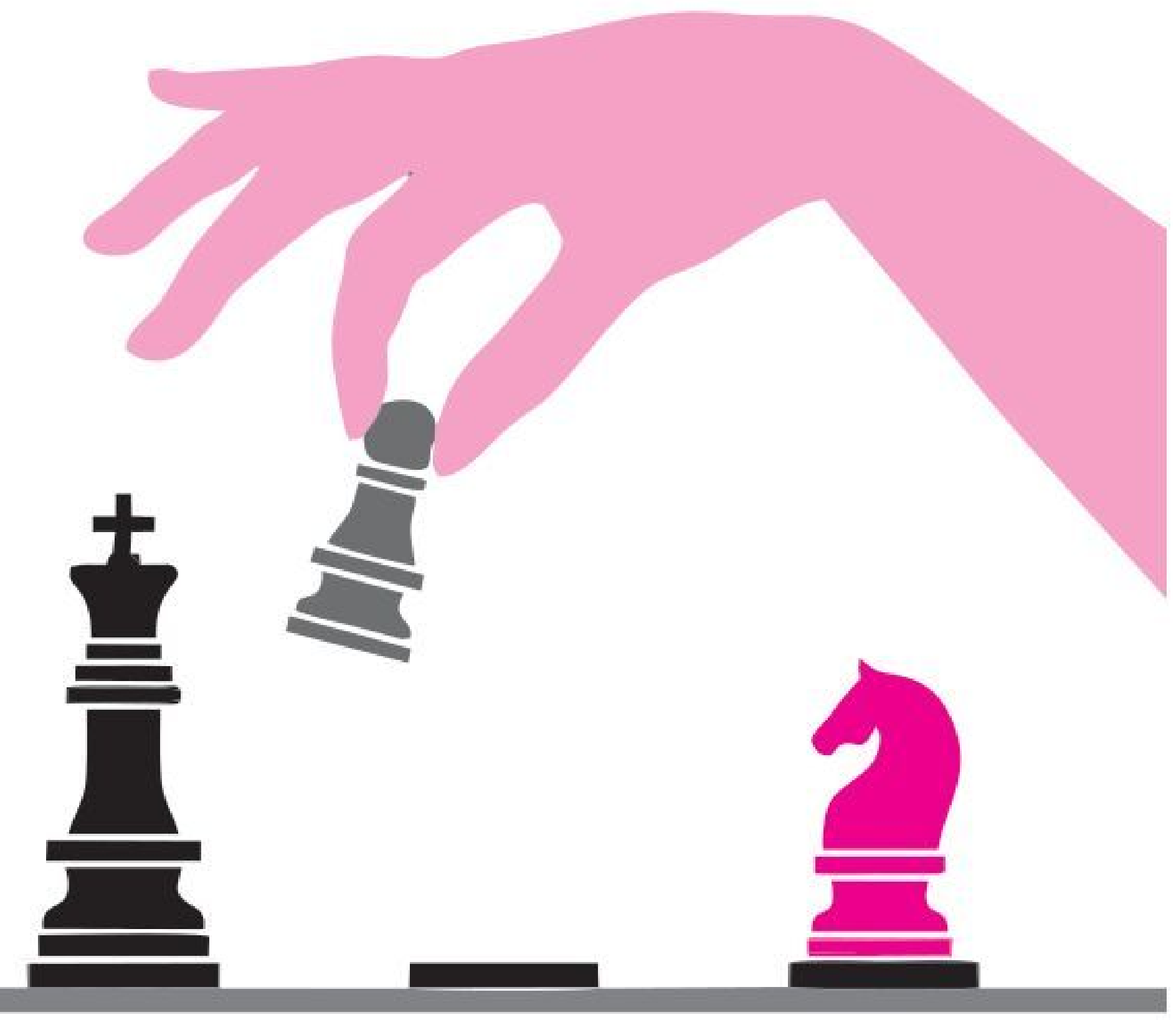




Single Option Correct Type

1. The sequence $<a_n>$ is defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n + 3, n \geq 1$. Then the value of $\lim\limits_{n\to\infty} \dfrac{\sqrt{a_n}+\sqrt{a_{4n}}+\sqrt{a_{4^2n}}+....+\sqrt{a_{4^{10}n}}}{\sqrt{a_n}+\sqrt{a_{2n}}+\sqrt{a_{2^2n}}+...+\sqrt{a_{2^{10}n}}}$ is

(a) 600 (b) 683 (c) 800 (d) 863

2. The function $f(x)=\begin{cases} e^{-\frac{1}{|x|}}\left(\sqrt{2}+\sin\dfrac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$

has a global minimum value at $x =$

(a) 0 (b) 1 (c) –1 (d) no where

3. Eight boxes, each a unit cube, are packed in a $2 \times 2 \times 2$ crate, open at the top. The boxes are taken out one by one, the number of ways this can be done is

(a) 252 (b) 2520 (c) 1256 (d) 1250

4. If $A=\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}=B^3+C^3$, where B and C are 2×2 matrices with integer elements then $Tr(B) + Tr(C) =$

(a) 1 (b) –1 (c) 3 (d) –3

5. It is given that the curve $y = |\sin x|$ intersects the line $y = kx (k > 0)$ at exactly three points $x = \alpha, \beta, \gamma$ where $\alpha < \beta < \gamma$. Then $\dfrac{\cos\gamma}{\sin\gamma+\sin 3\gamma}=$

(a) $\dfrac{1+\gamma^2}{4\gamma}$ (b) $\dfrac{\gamma^2-1}{4\gamma}$ (c) $\dfrac{4\gamma}{1+\gamma^2}$ (d) $\dfrac{4\gamma}{\gamma^2-1}$

6. Suppose a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$, $(a, b, c \in R)$ satisfies the following three conditions

(i) $f(x - 4) = f(2 - x)$ and $f(x) \geq x, \forall x \in R$

(ii) When $x\in(0,2), f(x)\leq\left(\dfrac{x+1}{2}\right)^2$

(iii) The minimum value of $f(x)$ on R is 0.

The possible value of $4(f(3) - f(2))$ is

(a) 4 (b) 5 (c) 6 (d) 7

7. For $\theta \in (0, \pi)$, which of the following statement is correct?

(a) $\int_0^\theta \sqrt{1+\cos^2 t}\, dt > \sqrt{\theta^2+\sin^2\theta}$

(b) $\int_0^\theta \sqrt{1+\cos^2 t}\, dt < \sqrt{\theta^2+\sin^2\theta}$

(c) $\int_0^\theta \sqrt{1+\cos^2 t}\, dt = \sqrt{\theta^2+\sin^2\theta}$

(d) $\int_0^\theta \sqrt{1+\cos^2 t}\, dt = \dfrac{1}{2}\sqrt{\theta^2+\sin^2\theta}$

8. A plane Π is given and on one side of the plane, lies three non-collinear points A, B, C such that the plane determined by them is not parallel to Π. Three random points A', B', C' in Π are selected. Let L, M, N be the midpoints of AA', BB' and CC' and let G be the centroid of ΔLMN. The locus of G as A', B', C' varies across Π is

(a) a plane parallel to Π

(b) a plane parpendicular to Π

(c) a line parallel to Π

(d) a line perpendicular to Π.

9. Let $f(x)=\sin^{-1}\left\{\dfrac{2g(x)}{1+\{g(x)\}^2}\right\}$, then

(a) $f(x)$ is decreasing, if $g(x)$ is increasing and $|g(x)| > 1$

(b) $f(x)$ is increasing, if $g(x)$ is increasing and $|g(x)| \leq 1$

(c) $f(x)$ is decreasing, if $g(x)$ is decreasing and $|g(x)| > 1$

(d) $f(x)$ is increasing, if $g(x)$ is decreasing and $|g(x)| \leq 1$

10. Which of the following functions are continuous on $(0, \pi)$?

By : Tapas Kr. Yogi (Prime Axis Classes), Visakhapatnam Mob : 09533632105, 09949428844

(a) $\tan x$ (b) $\int_0^x t\sin\left(\frac{1}{t}\right)dt$

(c) $\begin{cases} 1, & 0 < x \le \dfrac{3\pi}{4} \\ 3\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x\sin x, & 0 < x \le \pi/2 \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$

Comprehension Type

Paragraph for Q. No. 11 and 12

Let $\dfrac{b\cos x}{2\cos 2x - 1} = \dfrac{b + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x}$, $b \in R$

11. The equation has solutions if

(a) $b \in \left(-\infty, \frac{1}{2}\right) - \left\{-1, 0, \frac{1}{3}\right\}$

(b) $b \in (-\infty, 1) - \left\{-1, 0, \frac{1}{3}\right\}$

(c) $b \in R - \left\{-1, 0, \frac{1}{3}\right\}$

(d) none of these

12. For any value of b for which the equation has a solution, the number of solutions when $x \in (0, 2\pi)$ are always

(a) infinite

(b) depends upon the value of b

(c) four (d) none of these

Paragraph for Q. No. 13 and 14

A chess match between two players A and B is won by whoever first wins a total of 2 games. A's chances of winning, drawing or losing any particular game are a, b, c respectively. The games are independent and $a + b + c = 1$.

13. The probability that A wins match after $(n + 1)$ games $(n \ge 1)$ is

(a) $a^2(nb^{n-1} + n(n-1)c \cdot b^{n-2})$

(b) $a^2(n(n-1)b^{n-1} + n(n-2)c^n)$

(c) $c^2(nb^{n-1} + n(n-1) \cdot a \cdot b^{n-2})$

(d) $c^2(n(n-1)b^{n-1} + n(n-2)a^n)$

14. The probability that B wins the match after the 4th game is

(a) $3bc^2(b + 2a)$ (b) $3ac^2(b + 2a)$

(c) $3ac^2(b + 2a)$ (d) $3b^2c(b + 2a)$

Paragraph for Q. No. 15 and 16

The curve C has the equation $y = \dfrac{x}{\sqrt{x^2 - 2x + a}}$.

15. If $a > 1$, then C has

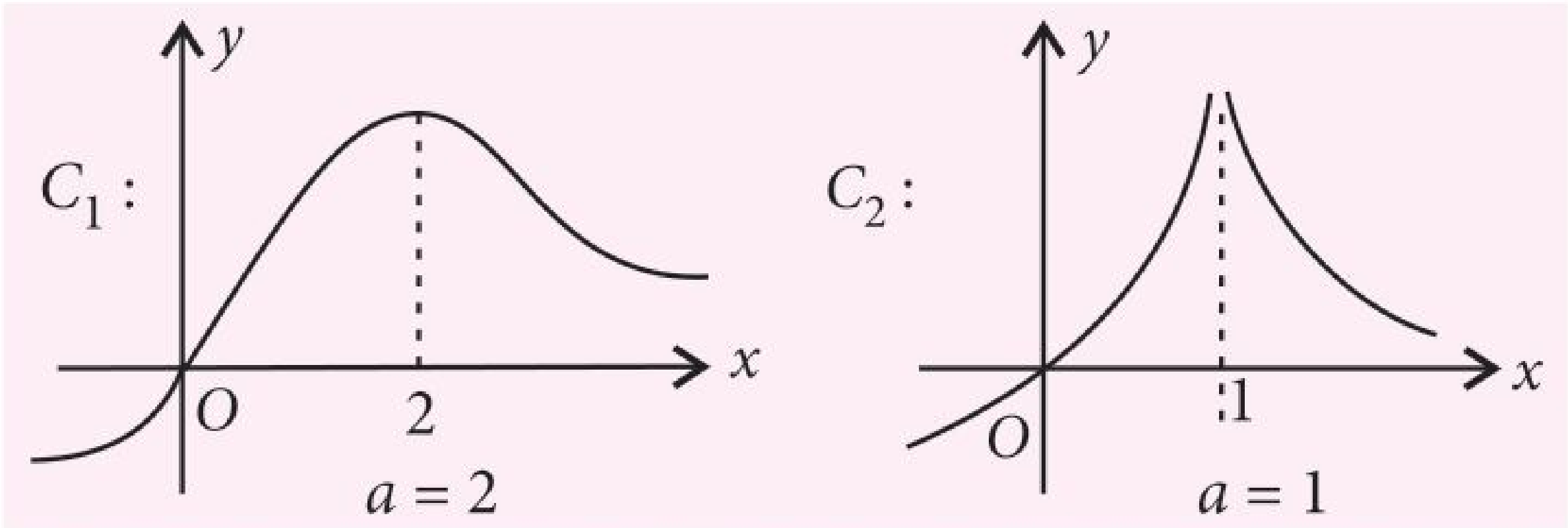
(a) exactly one stationary point

(b) exactly two stationary points

(c) no stationary point

(d) one or two stationary point depends on value of a.

16. Consider the two graphs C_1 and C_2 :



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CONCEPT MAP

TRIGONOMETRIC FUNCTIONS

Class XI

APPLICATION OF DERIVATIVES

Class XII

CONCEPT MAP

Relationship between Degree and Radian

- 1 radian = $\frac{180^\circ}{\pi}$ = $57^\circ 16'$ approximately
- $1^\circ = \left(\frac{\pi}{180}\right)$ radian = 0.01746 radian approximately

Sum and Difference of Two Angles

- $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
- $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$
- $\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta \cdot \tan\phi}$
- $\cot(\theta \pm \phi) = \frac{\cot\theta \cdot \cot\phi \mp 1}{\cot\phi \pm \cot\theta}$
- $\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
- $\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$
- $\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
- $\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$
- $2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$
- $2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi)$
- $2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi)$
- $2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$

Trigonometric Equations

- $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$
- $\cos\theta = 0 \Rightarrow \theta = (2n+1)\pi/2, n \in Z$
- $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in Z$
- $\cot\theta = 0 \Rightarrow \theta = (2n+1)\pi/2, n \in Z$
- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha, n \in Z$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in Z$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in Z$

Functions	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R-\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	R
$\sec\theta$	$R-\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	$R-(-1, 1)$
$\operatorname{cosec}\theta$	$R-\{n\pi, n \in Z\}$	$R-(-1, 1)$
$\cot\theta$	$R-\{n\pi, n \in Z\}$	R

Multiple Angles

- $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

  $= 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
- $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
- $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
- $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Properties of Triangles

- **Sine rule:** $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- **Cosine rule:**

  $a^2 = b^2 + c^2 - 2bc\cos A$, $b^2 = c^2 + a^2 - 2ca\cos B$,

  $c^2 = a^2 + b^2 - 2ab\cos C$
- **Napier's rule:**

  $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$,

  $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$,

  $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

Increasing and Decreasing Functions

- A function f is said to be an increasing function in (a, b), if $x_1 < x_2$

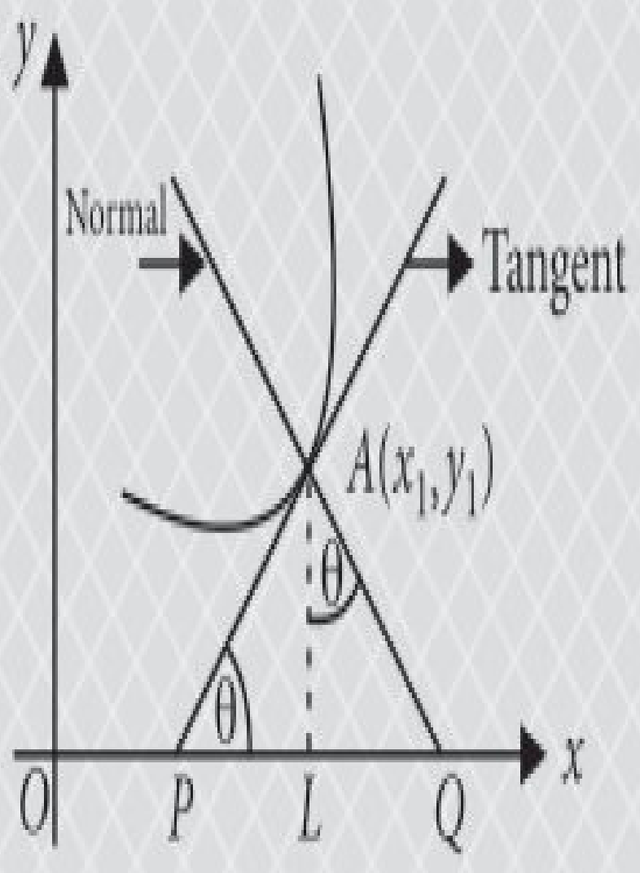
  $\Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a, b)$.
- A function f is said to be a decreasing function in (a, b), if $x_1 < x_2$

  $\Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in (a, b)$.
- $f(x)$ is known as non-decreasing if $f'(x) \geq 0$ and non-increasing if $f'(x) \leq 0$.
- A function f is said to be monotonic in an interval if it is either increasing or decreasing in that interval.

Tangents and Normals

- Equation of tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}(x - x_1)$
- Equation of normal to the curve $y = f(x)$ at point $P(x_1, y_1)$ is $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}(x - x_1)$

Length of Tangents and Normals



- Length of tangent = AP

  $= \left|\frac{y_1\sqrt{1+\left(\frac{dy_1}{dx_1}\right)^2}}{\frac{dy_1}{dx_1}}\right|$
- Length of normal = $AQ = \left|y_1\sqrt{1+\left(\frac{dy_1}{dx_1}\right)^2}\right|$
- Length of sub-tangent $= \left|\frac{y_1}{\frac{dy_1}{dx_1}}\right|$
- Length of sub-normal $= \left|y_1\frac{dy_1}{dx_1}\right|$

Rate of Change of Quantities

- Let $y = f(x)$, then $\frac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y w.r.t. x.

Errors and Approximations

Let $y = f(x)$, Δx be the small change in x and Δy be the corresponding change in y. Then, $\Delta y = \frac{dy}{dx}(\Delta x)$

- **Absolute Error :** Δx
- **Relative Error :** $\Delta x/x$
- **Percentage Error :** $\left(\frac{\Delta x}{x}\times 100\right)$

Maxima and Minima

- **Local Maxima :** At $x = a$; $f(a-h) < f(a) > f(a+h)\ \forall\ h \to 0^+$
- **Local Minima :** At $x = a$; $f(a-h) > f(a) < f(a+h)\ \forall\ h \to 0^+$
- **Extremum :** At $x = a$; $f(x)$ is said to have extremum if it has either maximum or minimum.

First Derivative Test

- $x = a$ be the point of local maxima if $f'(a) = 0$ and $f'(x)$ changes its sign from +ve to –ve.
- $x = a$ be the point of local minima if $f'(a) = 0$ and $f'(x)$ changes its sign from –ve to +ve.

Second Derivative Test

- Find the roots of $f'(x) = 0$.
- Suppose $x = a$ is one of the roots.
- At $x = a$, if $f''(x) < 0 \Rightarrow f(x)$ will attain local maximum value, if $f''(x) > 0 \Rightarrow f(x)$ will attain local minimum value.

Higher Order Derivative Test

- If $f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0$ and $f^n(a)$ is a finite number other than zero, then
  - If n is even and $f^n(a) < 0 \Rightarrow x = a$ is a point of local maxima.
  - If n is even and $f^n(a) > 0 \Rightarrow x = a$ is a point of local minima.
  - If n is odd $\Rightarrow x = a$ is neither a point of maxima nor a point of minima.

(a) Both C_1, C_2 are correct graphs
(b) Only C_1 is correct graph
(c) Only C_2 is correct graph
(d) Neither C_1 nor C_2 are correct graphs

NUMERICAL VALUE TYPE

17. The possible values of $\sqrt{x^2+x+1}-\sqrt{x^2-x+1}$ as x ranges over all real numbers is an interval (a, b), then $|b - a| =$ ______.

18. The number of real solutions of the equation $\cos(\cos(\cos(\cos x))) = \sin(\sin(\sin(\sin x)))$ is/are ______.

19. Let $f : (0, \infty) \rightarrow R$ be a twice continuously differentiable function such that $|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leq 1$ for all x, then $\lim\limits_{x\to\infty} f(x)=$ ______.

20. Given $I_n = \int\limits_0^1 (1+x+x^2+...+x^{n-1})\,(1+3x+5x^2+ \;.... +(2x-1)x^{n-1})dx$, then $\sqrt{I_9} =$ ______.

SOLUTIONS

1. (b) : Using the recurrence relation,

$a_k = a_{k-1} + 4(k - 1) + 3 = a_{k-2} + 4(k - 2) + 4(k - 1) + 2 \cdot 3$

$= a_1 + 4(1 + 2 + 3 + ... + k - 1) + (k - 1) \cdot 3$

$= (2k + 3)(k - 1)$

$$\text{So, } \lim_{n\to\infty}\frac{\sqrt{a_{kn}}}{n} = \lim_{n\to\infty}\sqrt{\left(2k+\frac{3}{n}\right)\left(k-\frac{1}{n}\right)} = \sqrt{2}\,k$$

$$\text{So, required limit} = \frac{1+4+4^2+...+4^{10}}{1+2+2^2+...+2^{10}} = \frac{4^{11}-1}{3(2^{11}-1)}$$

$$= \frac{2^{11}+1}{3} = 683$$

2. (a) : We have,

$$f'(x) = \begin{cases} \dfrac{1}{x^2}e^{\frac{-1}{|x|}}\left(\left(\sqrt{2}+\sin\dfrac{1}{x}\right)\text{sgn}(x)-\cos\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Since, $\left|\sin\frac{1}{x} \pm \cos\frac{1}{x}\right| \leq \sqrt{2}$, we have

$f'(x) \geq 0$ if $x > 0$ and $f'(x) \leq 0$ if $x < 0$ and $f'(0) = 0$. So, global minimum at $x = 0$, because $f(x) > f(0) = 0$ for $x \neq 0$.

3. (b) : If number of cubes are T_1, T_2, T_3, T_4 on top layer and B_1, B_2, B_3, B_4 on bottom layer, then only restriction is no B_i can come out before corresponding T_i.

So, total possibilities $= \dfrac{8!}{2!2!2!2!} = 2520$

4. (d) : We have, $A^2 + 3A + 2I = O$

$\Rightarrow \quad A = (A + I)^3 + (-I)^3$

i.e., $B = A + I$ and $C = -I$

i.e., $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

So, $Tr(B) + Tr(C) = -1 - 2 = -3$

5. (a) : $y = |\sin x|$ and $y = kx$ intersects at $\alpha = 0$, $\beta \in (0, \pi)$ and $\gamma \in \left(\pi, \frac{3\pi}{2}\right)$. So, at $(\gamma, -\sin\gamma)$ we have,

$$\left.\frac{dy}{dx}\right|_{x=\gamma} = -\cos\gamma = \frac{-\sin\gamma - 0}{\gamma - 0} \;\Rightarrow\; \tan\gamma = \gamma$$

$$\text{Hence, } \frac{\cos\gamma}{\sin\gamma + \sin 3\gamma} = \frac{1+\tan^2\gamma}{4\tan\gamma} = \frac{1+\gamma^2}{4\gamma}$$

6. (d) : From condition (iii), $a > 0$ and from condition (i), $x = -1$ is axis of symmetry. So, let $f(x) = a(x + 1)^2$. Also, from (i), $f(1) \geq 1$ but from (ii), $f(1) \leq 1$

$\Rightarrow \quad f(1) = 1$ So, $a = \frac{1}{4}$. Hence, $f(x) = \frac{1}{4}(x+1)^2$

$\therefore \quad 4(f(3) - f(2)) = 7$

7. (a) : Notice that $\int\limits_0^\theta \sqrt{1+\cos^2 t}\,dt$ is the length of the curve $y = \sin x$ from $(0, 0)$ to $(\theta, \sin\theta)$ and $\sqrt{\theta^2+\sin^2\theta}$ is the distance between these points.

8. (a) : Let $d(p)$ denote the distance of a point P from Π, with P being on the same side of Π as A, B, C. Let G_1 be the fixed centroid of ΔABC and G_1' be the centroid of $\Delta A'B'C'$. It is obvious that G is the midpoint of G_1G_1'. Hence varying G_1' across Π, we have the locus of G as the plane Π' parallel to Π such that $P \in \Pi'$,

$$d(P) = \frac{d(G_1)}{2} = \frac{d(A)+d(B)+d(C)}{6}$$

9. (b) : $f(x) = \sin^{-1}(\sin 2\theta)$, where $g(x) = \tan\theta$

$= 2\theta$, $|g(x)| \leq 1$

$= 2\tan^{-1}[g(x)], |g(x)| \leq 1$

When, $|g(x)| > 1$, $f(x) = \sin^{-1}\left\{\dfrac{2/g(x)}{1+[1/g(x)]^2}\right\}$

$= \sin^{-1}(\sin 2\theta)$, where $\tan\theta = \dfrac{1}{g(x)}$

$= 2\theta = 2\cot^{-1}[g(x)],\ |g(x)| > 1$

$$\therefore f'(x) = \begin{cases} \dfrac{2}{1+[g(x)]^2}\cdot g'(x), & |g(x)| \le 1 \\ \dfrac{-2}{1+[g(x)]^2}\cdot g'(x), & |g(x)| > 1 \end{cases}$$

10. (b) : The function $f(x) = \tan x$ is not defined at $x = \frac{\pi}{2}$, so $f(x)$ is not continuous on $(0, \pi)$.

For the function $f(x) = x \sin\frac{1}{x}$ is continuous on $(0, \pi)$ and the integral function of a continuous function is continuous, therefore $F(x) = \int_0^x t\sin\frac{1}{t}\,dt$ is continuous on $(0, \pi)$.

$$\text{For the function } f(x) = \begin{cases} 1, & 0 < x \le \dfrac{3\pi}{4} \\ 3\sin\dfrac{2}{9}x, & \dfrac{3\pi}{4} < x < \pi \end{cases}$$

$$\text{Now, } \lim_{x\to\frac{3\pi}{4}^+} f(x) = \lim_{h\to 0} f\left(\frac{3\pi}{4}+h\right)$$

$$= \lim_{h\to 0} 3\sin\frac{2}{9}\left(\frac{3\pi}{4}+h\right)$$

$= 3\sin\pi/6 = \frac{3}{2}$ and $f(3\pi/4) = 1$

$\therefore$ $f(x)$ is discontinuous at $x = 3\pi/4$

$\Rightarrow$ $f(x)$ is discontinuous on $(0, \pi)$

$$\text{For the function } F(x) = \begin{cases} x\sin x, & 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi + x), & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$F(\pi/2) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{Now, } \lim_{x\to\frac{\pi}{2}^-} F(x) = \lim_{h\to 0} F\left(\frac{\pi}{2}-h\right)$$

$$= \lim_{h\to 0}\left(\frac{\pi}{2}-h\right)\sin\left(\frac{\pi}{2}-h\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$\lim_{x\to\frac{\pi}{2}^+} F(x) = \lim_{h\to 0} F\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0}\frac{\pi}{2}\sin\left(\frac{3\pi}{2}+h\right)$$

$$= \frac{\pi}{2}\sin\frac{3\pi}{2} = -\frac{\pi}{2}$$

$\therefore$ $F(x)$ is not continuous at $x = \frac{\pi}{2}$.

(11-12) : We have, $\dfrac{b\cos x}{2\cos 2x - 1} = \dfrac{b + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x}$ ...(i)

(i) $2\cos 2x - 1 \ne 0 \Rightarrow x \ne n\pi \pm \frac{\pi}{6}$

(ii) $\tan x \ne 0 \Rightarrow x \ne n\pi$

(ii) $\cos^2 x - 3\sin^2 x \ne 0 \Rightarrow x \ne n\pi \pm \frac{\pi}{6}$

Also, $2\cos 2x - 1 = 2(\cos^2 x - \sin^2 x) - (\cos^2 x + \sin^2 x)$

$= \cos^2 x - 3\sin^2 x$

$\therefore$ (i) reduces to $b\sin x = b + \sin x \Rightarrow \sin x = \frac{b}{b-1}$

Now, $-1 \le \sin x \le 1$

$$\Rightarrow -1 \le \frac{b}{b-1} \le 1 \Rightarrow \frac{b}{b-1} + 1 \ge 0 \text{ and } \frac{b}{b-1} - 1 \le 0$$

$$\Rightarrow \frac{2b-1}{b-1} \ge 0 \text{ and } \frac{1}{b-1} \le 0$$

$$\Rightarrow b \in \left(-\infty, \frac{1}{2}\right] \cup (1, \infty) \text{ and } b \in (-\infty, 1)$$

Now, $\sin x \ne 0, \pm\frac{1}{2}$ or $b \ne 0, -1, \frac{1}{3}$

When $b = \frac{1}{2}$, then $\sin x = -1$, which is not possible, as $\tan x$ is not defined for this value of $\sin x$.

So, $b \in \left(-\infty, \frac{1}{2}\right) - \left\{-1, 0, \frac{1}{3}\right\}$

For any other value of b, $\sin x$ takes two values for $x \in (0, \pi)$. Hence, four solutions are there for $x \in (0, 2\pi)$.

11. (a) **12. (c)**

(13-14) : A can win after the $(n + 1)^{\text{th}}$ game in the following 2 ways.

(i) A wins exactly one of the first n games, draws $(n - 1)$ games and wins the $(n + 1)^{\text{th}}$ game.

(ii) A loses exactly one of the first n games, wins exactly one of the first n games and draws $(n - 2)$ games and finally wins the $(n + 1)^{\text{th}}$ game.

For (i), probability = $({}^nP_1 \cdot ab^{n-1})a$ and for (ii) the probability is $({}^nP_2)\cdot ac\cdot b^{n-2}\cdot a$

So, required probability

$= na^2b^{n-1} + n(n-1)a^2cb^{n-2}$

$= na^2[b^{n-1} + (n-1)\cdot c\cdot b^{n-2}]$

Now, for B winning, we put $n = 3$ and interchange a and c in the above equation.

So required probability $= 3c^2[b^2 + 2ab] = 3bc^2[b + 2a]$

13. (a) **14. (a)**

(15-16) : $\frac{dy}{dx} = \frac{(a-x)}{(x^2-2x+a)^{3/2}} = 0$ only at $x = a$

Exactly one stationary point.

And for graph of C_1, $a = 2$, has stationary point at $(2, \sqrt{2})$ and $x \to \infty$, $y \to 1$ and $x \to -\infty$, $y \to -1$.

For $a = 1$, $y = \frac{x}{|x-1|}$, a vertical asymptote at $x = 1$.

15. (a) **16. (a)**

17. (2) : Let $A = (x, 0)$, $B = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$, $C = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Then the given expression is $AB - AC$

So, from triangle inequality, $|AB - AC| < BC$ and $BC = 1$. So, given expression $\in (-1, 1)$.

Hence, $|b - a| = 2$

18. (0) : Let $f(x) = \sin(\sin x)$ and $g(x) = \cos(\cos x)$

Then $g(x) - f(x)$

$$= 2\cos\left(\frac{\pi}{4} - \frac{\cos x - \sin x}{2}\right)\cdot \sin\left(\frac{\pi}{4} - \frac{\cos x + \sin x}{2}\right)$$

Now, $\left|\frac{\cos x \pm \sin x}{2}\right| = \frac{\sqrt{2}}{2}\left|\sin\left(x \pm \frac{\pi}{4}\right)\right| < \frac{\pi}{4}$

$\Rightarrow$ $g(x) - f(x) > 0$ and $f(x)$ is increasing

So, $f(f(x)) < f(g(x)) < g(g(x))$

Hence, no solution.

19. (0) : $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{e^{x^2/2}\cdot f(x)}{e^{x^2/2}}$

Now applying L-Hospital's Rule twice, we get

$$\lim_{x\to\infty} \frac{f''(x) + 2xf'(x) + (x^2+1)f(x)}{x^2+1} = 0$$

20. (9) : Let $a = 1 + x + x^2 + \dots + x^{n-1}$ and $b = 1 + 3x + 5x^2 + \dots + (2n-1)x^{n-1}$

Then $I_n = \int_0^1 (ab)\,dx$

So, $I_{n+1} = \int_0^1 (a + x^n)(b + (2n+1)x^n)dx$

$$= I_n + (2n+1)\int_0^1 ax^n dx + \int_0^1 bx^n dx + \int_0^1 (2n+1)\,x^{2n}dx$$

On simplification, $I_{n+1} = I_n + 1 + 2n$

So, $I_9 = I_8 + 1 + 2(8)$

$I_8 = I_7 + 1 + 2(7)$

....

....

$I_2 = I_1 + 1 + 2(1)$

Adding all we get, $I_9 = I_1 + 80$ and $I_1 = \int_0^1 (1)(1)dx = 1$

So, $I_9 = 81 \Rightarrow \sqrt{I_9} = 9$

◆◆

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MOCK TEST PAPER

ISI

Indian Statistical Institute

Exam on 18th July, 2021

*ALOK KUMAR

PART A

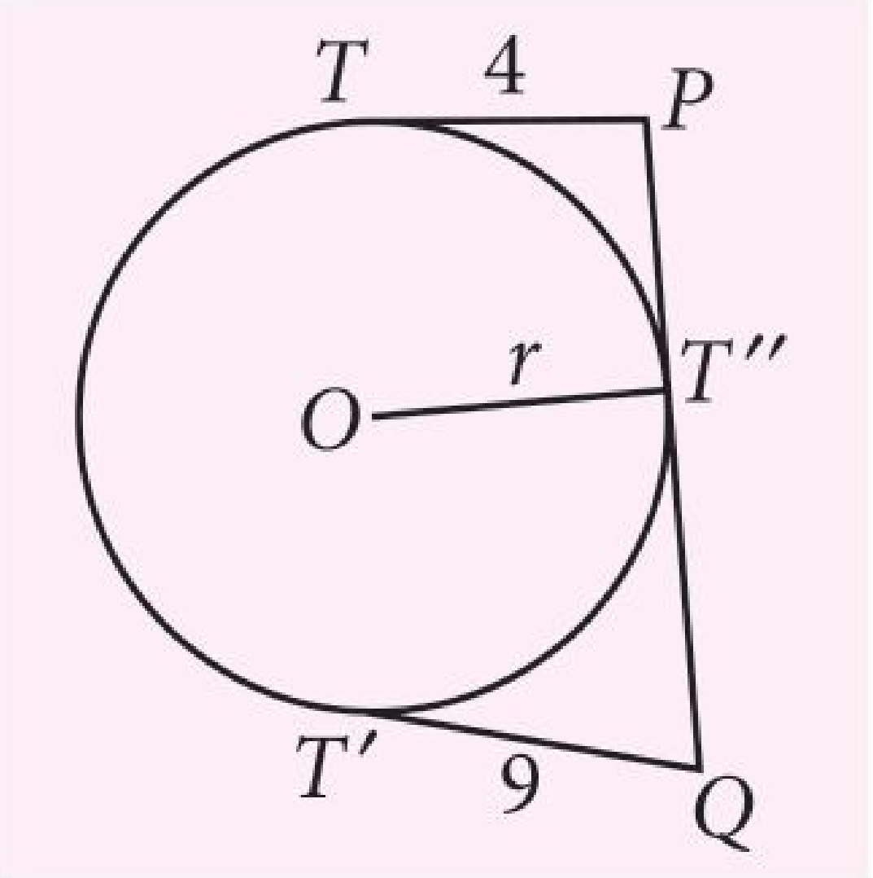
1. A circle of radius r is inscribed in a right isosceles triangle, and a circle of radius R is circumscribed about the triangle. Then R/r equals

(a) $1+\sqrt{2}$ (b) $\frac{2+\sqrt{2}}{2}$

(c) $\frac{\sqrt{2}-1}{2}$ (d) $\frac{1+\sqrt{2}}{2}$

(e) $2(2-\sqrt{2})$

2. In the adjoining figure TP and $T'Q$ are parallel tangents to a circle of radius r, with T and T' the points of tangency. $PT''Q$ is a third tangent with T'' as point of tangency. If $TP = 4$ and $T'Q = 9$, then r is

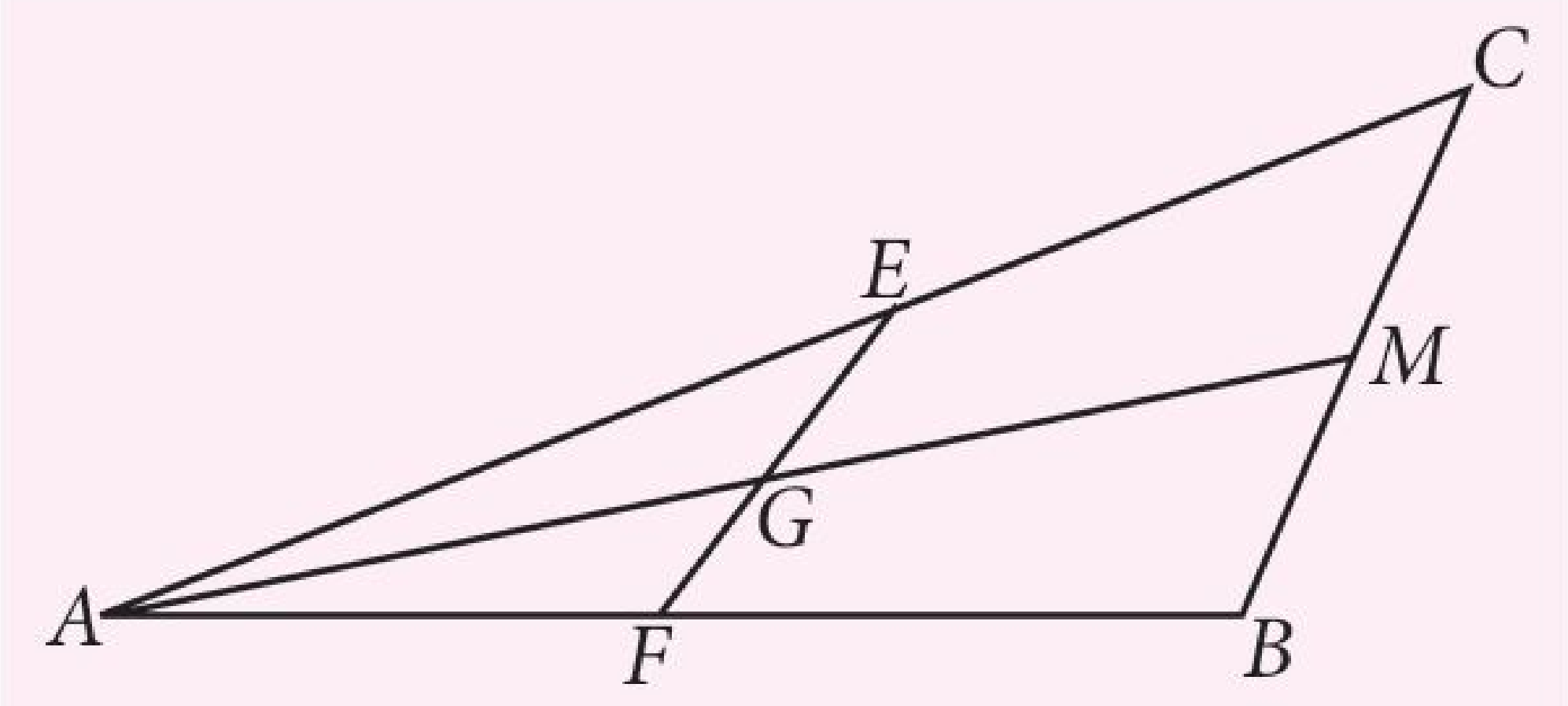


(a) 25/6 (b) 6 (c) 25/4

(d) a number other than 25/6, 6, 25/4

(e) not determinable from the given information

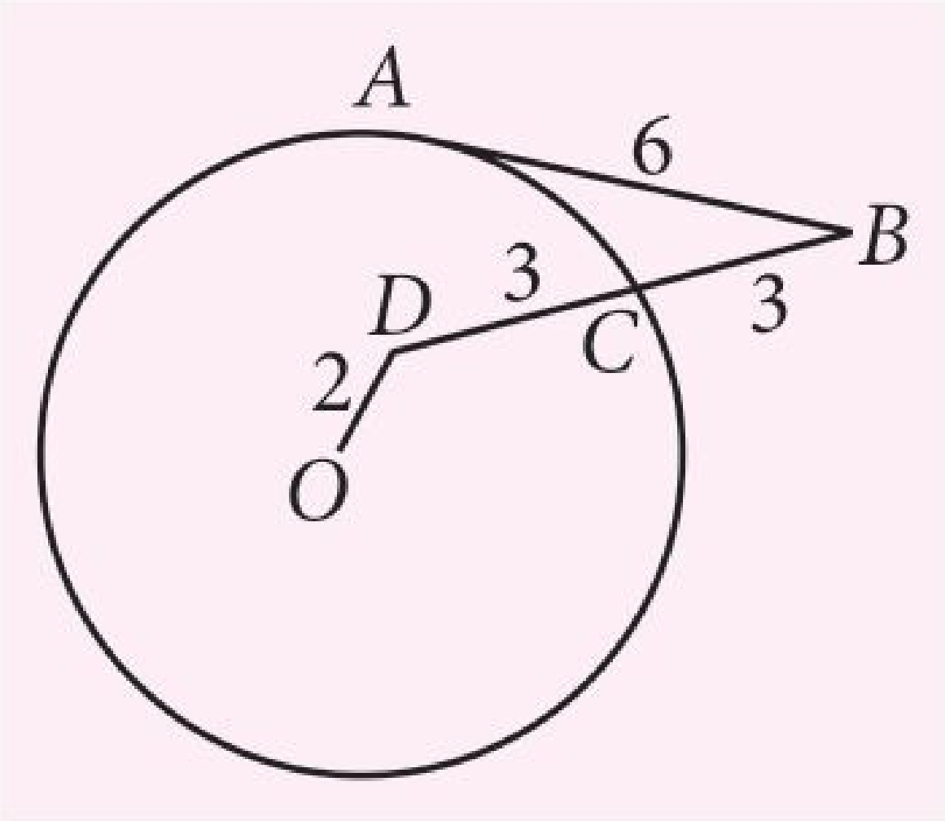
3. In triangle ABC shown in the adjoining figure, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$, then $\frac{EG}{GF}$ equals



(a) 3/2 (b) 4/3 (c) 5/4 (d) 6/5

(e) Not enough information given to solve the problem

4. In the adjoining figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3$, $OD = 2$ and $AB = 6$, then the radius of the circle is



(a) $3+\sqrt{3}$ (b) $\frac{15}{\pi}$ (c) $\frac{9}{2}$ (d) $2\sqrt{6}$

(e) $\sqrt{22}$

5. Let a, b, c and d be the lengths of sides MN, NP, PQ and QM, respectively, of quadrilateral $MNPQ$. If A is the area of $MNPQ$, then

(a) $A=\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a convex

(b) $A=\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle

(c) $A\leq\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a rectangle

(d) $A\leq\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

(e) $A\geq\left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if $MNPQ$ is a parallelogram

---

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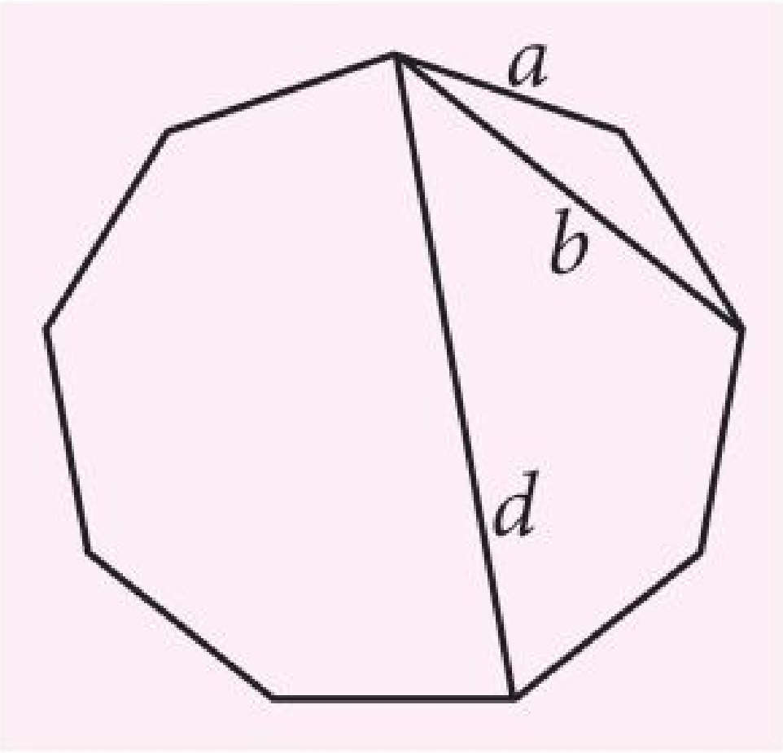
He trains IIT and Olympiad aspirants.

6. If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal respectively, of a regular nonagon (see adjoining figure), then



(a) $d = a + b$ (b) $d^2 = a^2 + b^2$

(c) $d^2 = a^2 + ab + b^2$ (d) $b=\frac{a+d}{2}$

(e) $b^2 = ad$

7. Sides AB, BC, CD and DA respectively, of convex quadrilateral $ABCD$ are extended past B, C, D and A to points B', C', D' and A'. Also, $AB = BB' = 6$, $BC = CC' = 7$, $CD = DD' = 8$ and $DA = AA' = 9$; and the area of $ABCD$ is 10. The area of $A'B'C'D'$ is

(a) 20 (b) 40 (c) 45 (d) 50

(e) 60

8. The function f satisfies the functional equation

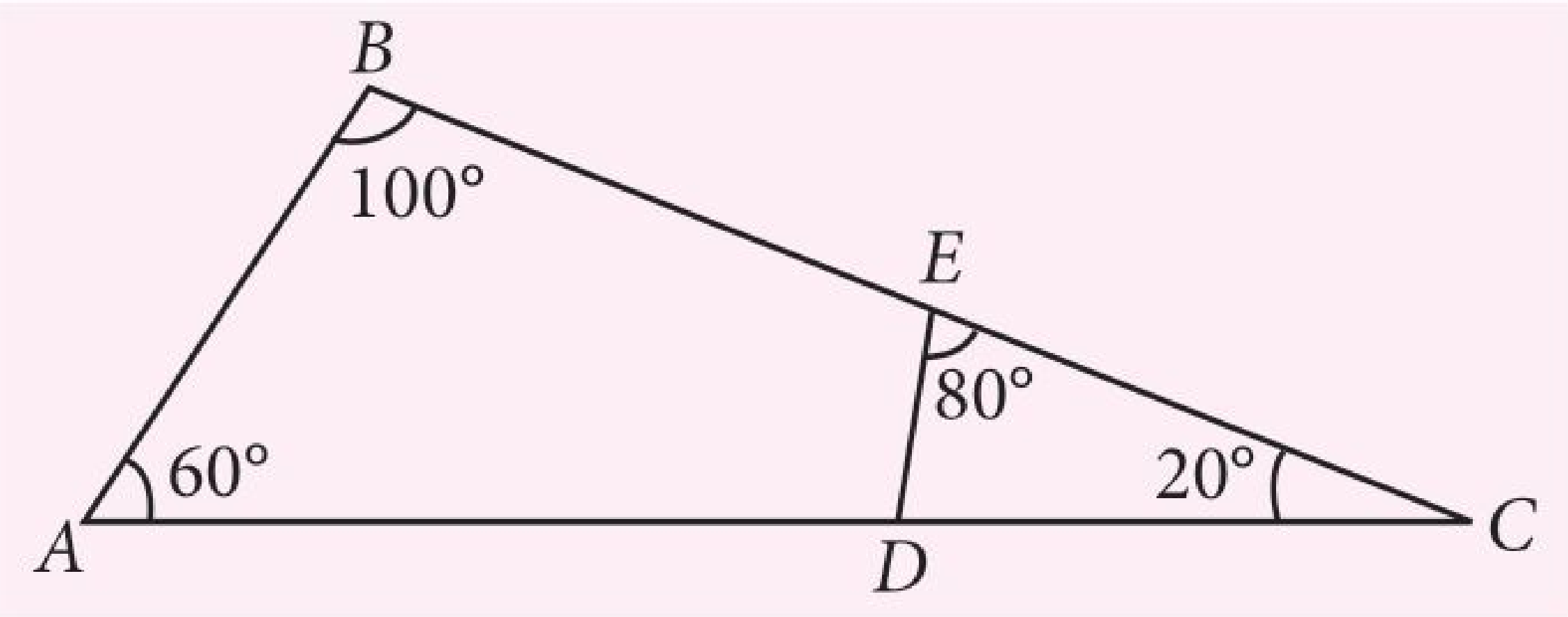
$$f(x) + f(y) = f(x + y) - xy - 1$$

for every pair x, y of real number. If $f(1) = 1$, then the number of integers $n \neq 1$ for which $f(n) = n$ is

(a) 0 (b) 1 (c) 2 (d) 3

(e) Infinite

9. In ΔABC, E is the midpoint of side BC and D is on side AC. If the length of AC is 1 and $\angle BAC = 60°$, $\angle ABC = 100°$, $\angle ACB = 20°$ and $\angle DEC = 80°$, then the area of ΔABC plus twice the area of ΔCDE equals



(a) $\frac{1}{4}\cos 10°$ (b) $\frac{\sqrt{3}}{8}$

(c) $\frac{1}{4}\cos 40°$ (d) $\frac{1}{4}\cos 50°$

(e) $\frac{1}{8}$

10. If $b > 1$, $\sin x > 0$, $\cos x > 0$ and $\log_b \sin x = a$, then $\log_b \cos x$ equals

(a) $2\log_b\left(\frac{a}{1-b^2}\right)$ (b) $\sqrt{1-a^2}$

(c) b^{a^2} (d) $\frac{1}{2}\log_b(1-b^{2a})$

(e) None of these

11. In the non-decreasing sequence of odd integers $\{a_1, a_2, a_3, ...\} = \{1, 3, 3, 3, 5, 5, 5, 5, 5, ...\}$ each positive odd integer k appears k times. It is a fact that there are integers b, c and d such that, for all positive integers n, $a_n = b[\sqrt{n+c}]+d$, where $[x]$ denotes the largest integer not exceeding x. The sum $b + c + d$ equals

(a) 0 (b) 1 (c) 2 (d) 3

(e) 4

12. How many ordered triples (x, y, z) of integers satisfy the system of equations below?

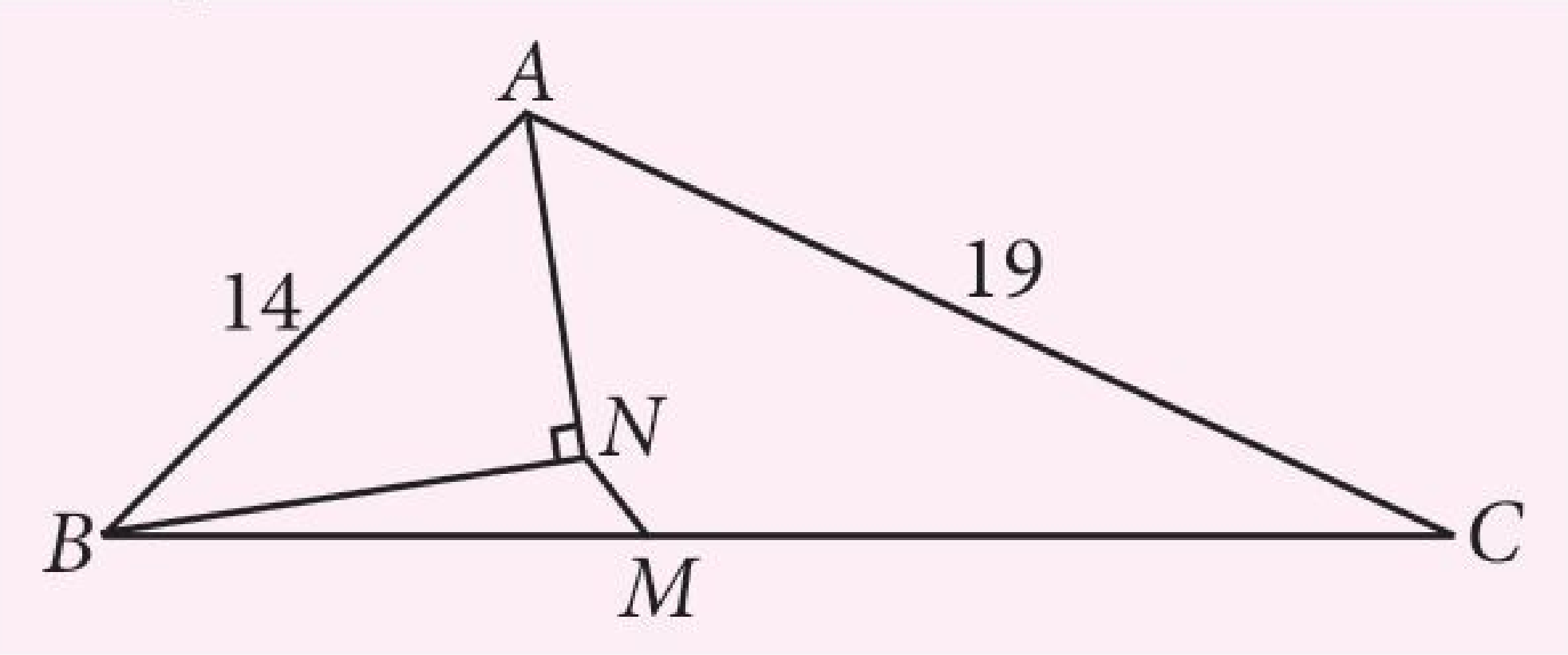
$$x^2 - 3xy + 2y^2 - z^2 = 31,$$
$$-x^2 + 6yz + 2z^2 = 44,\ x^2 + xy + 8z^2 = 100.$$

(a) 0 (b) 1 (c) 2

(d) A finite number greater than two

(e) Infinitely many

13. In ΔABC, M is the midpoint of side BC, AN bisects $\angle BAC$, $BN \perp AN$ and θ is the measure of $\angle BAC$. If sides AB and AC have lengths 14 and 19 respectively, then length MN equals



(a) 2 (b) 5/2

(c) $\frac{5}{2}-\sin\theta$ (d) $\frac{5}{2}-\frac{1}{2}\sin\theta$

(e) $\frac{5}{2}-\frac{1}{2}\sin\left(\frac{\theta}{2}\right)$

14. Alice, Bob and Carol repeatedly take turns tossing a die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first one to toss a six. (The probability of obtaining a six on any toss is 1/6, independent of the outcome of any other toss.)

(a) 1/3 (b) 2/9 (c) 5/18 (d) 25/91

(e) 36/91

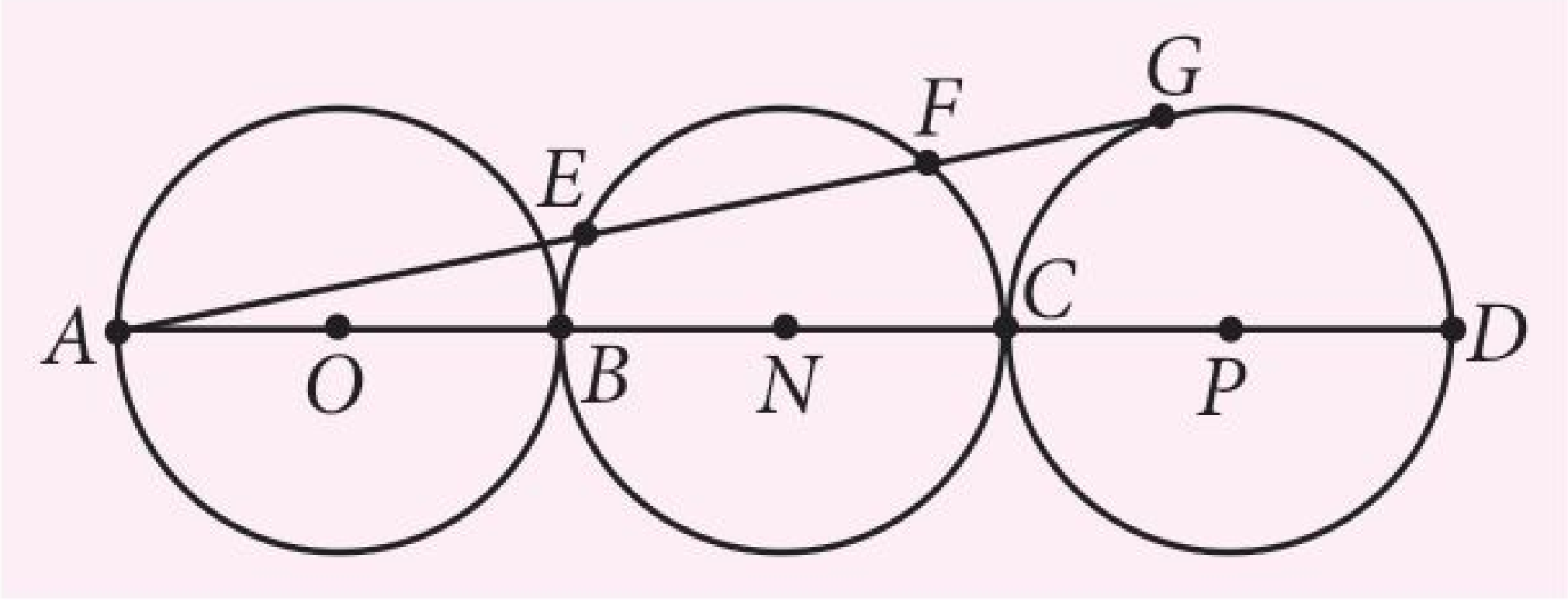
15. Find the units digit in the decimal expansion of

$$(15+\sqrt{220})^{19}+(15+\sqrt{220})^{82}.$$

(a) 0 (b) 2 (c) 5 (d) 9

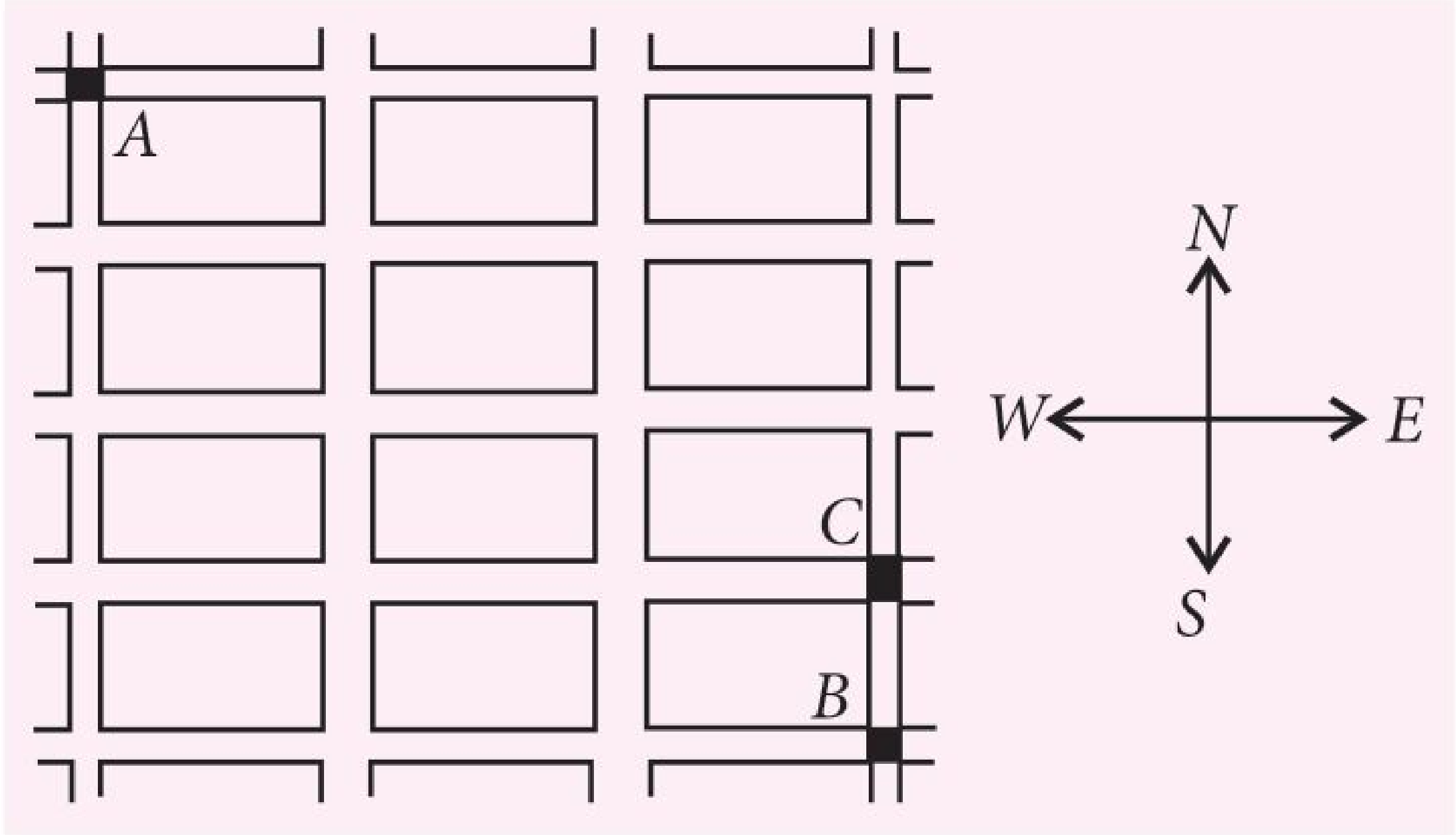
(e) None of these

16. In the given figure, points B and C lie on line segment AD, and AB, BC and CD are diameters of circles O, N and P, respectively. Circles O, N and P all have radius 15, and the line AG is tangent to circle P at G. If AG intersects circle N at points E and F, then chord EF has length



(a) 20 (b) $15\sqrt{2}$ (c) 24 (d) 25
(e) None of these

17. The given figure is a map of part of a city: the small rectangles are blocks and the spaces in between are streets. Each morning a student walks from intersection A to intersection B, always walking along streets shown, always going east or south. For variety, at each intersection where he has a choice, he chooses with probability 1/2 (independent of all other choices) whether to go east or south. Find the probability that, on any given morning, he walks through intersection C.



(a) $\frac{11}{32}$ (b) $\frac{1}{2}$ (c) $\frac{4}{7}$ (d) $\frac{21}{32}$
(e) $\frac{3}{4}$

18. Suppose $z = a + ib$ is a solution of the polynomial equation $c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0 = 0$,
where c_0, c_1, c_2, c_3, c_4, a and b are real constants and $i^2 = -1$. Which one of the following must also be a solution ?
(a) $-a - ib$ (b) $a - ib$
(c) $-a + ib$ (d) $b + ia$
(e) None of these

PART B

1. Show that if n is congruent to 2 or 3 modulo 4, then is it not possible to get a rearrangement $(x_1, x_2, ..., x_n)$ of $(1, 2, ..., n)$ such that all are distinct?

2. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer m such that $p(n)p(n + 1) = p(m)$.

3. Let AC and BD be two chords of a circle with center O such that they intersect at right-angles inside the circle at the point M. Suppose K and L are the mid-points of the chords AB and CD respectively. Prove that $OKML$ is a parallelogram.

4. Find a finite sequence of 16 numbers such that
(a) it reads the same from left to right as from right to left,
(b) the sum of any 7 consecutive terms is -1 and
(c) the sum of any 11 consecutive terms is $+ 1$.

5. I have six friends and during a certain vacation, I met them during several dinners. I found that I dined with all the six exactly on one day; with every five of them on 2 days; with every four on 3 days; with every three on 4 days : with every two on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

6. In an examination 70% students of a class failed in Tamil, 75% in English, 80% in Mathematics and 85% in Science. What is the least percentage of the students who failed in all four subjects?

7. Determine with proof, all the positive integers n for which
(i) n is not the square of any integer and
(ii) $[\sqrt{n}]^3$ divides n^2.

($[x]$ denotes the largest integer that is less than or equal to x)

8. Solve for integers x, y, z:
$x + y = 1 - z,\ x^3 + y^3 = 1 - z^2.$

9. Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the midpoint of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the midpoint of XZ.

10. A triangle ABC has incentre I. Points X, Y are located on the line segments AB, AC respectively so that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that X, I, Y are collinear, find the possible values of the measure of angle A.

11. Let A be a subset of $\{1, 2, 3, ..., 2n - 1, 2n\}$ containing $n + 1$ elements. Show that

(i) Some two elements of A are relatively prime

(ii) Some two elements of A have the property that one divides the other.

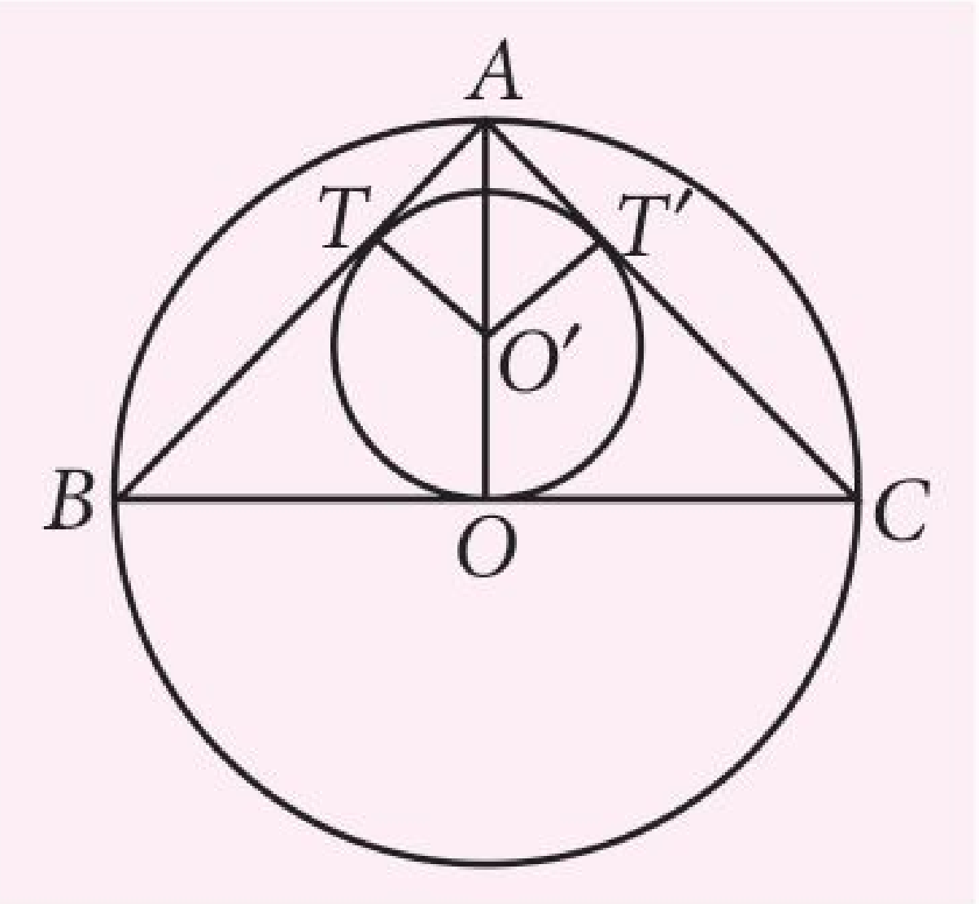
12. In a class of 25 students, there are 17 cyclists, 13 swimmers and 8 weight lifters and no one is all the three. In a certain mathematics examination 6 students got grades D or E. If the cyclists, swimmers and weight lifters all got grade B or C, determine the number of students who got grade A. Also, find the number of cyclists who are swimmers.

SOLUTIONS

PART A

1. (a) : In the adjoining figure, ΔABC is a right isosceles triangle, with $\angle BAC = 90°$ and $AB = AC$, inscribed in a circle with center O and radius R. The line segment AO has length R and bisects line segment BC and $\angle BAC$. A circle with center O' lying on AO and radius r is inscribed in ΔABC. The sides AB and AC are tangent to the inscribed circle with points of tangency T and T', respectively. Since $\Delta ATO'$ has angles 45° – 45° – 90° and $O'T = r$, we have $AT = r$ and $O'A = R - r = r\sqrt{2}$.

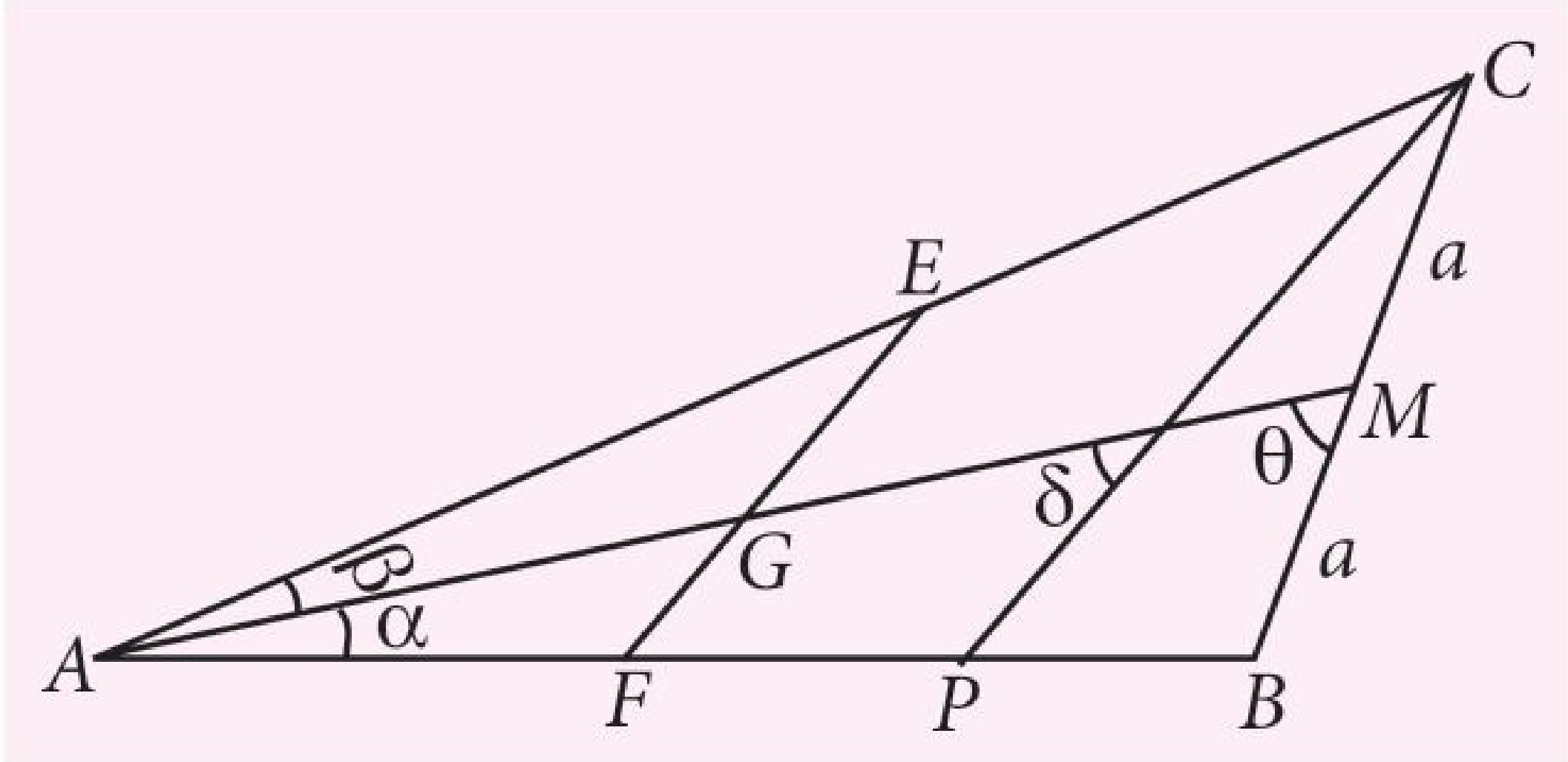
Hence $R = r + r\sqrt{2}$ and $\dfrac{R}{r} = 1 + \sqrt{2}$



2. (b) : Since $TP = T''P$, $OT = OT'' = r$, and $\angle PTO = \angle PT''O = 90°$, we have $\Delta OTP \cong \Delta OT''P$. Similarly $\Delta OT'Q \cong \Delta OT''Q$. Letting $x = \angle TOP = \angle POT''$ and $y = \angle T''OQ = \angle QOT'$, we obtain $2x + 2y = 180°$. But this implies that $\angle POQ = x + y = 90°$. Therefore ΔPOQ is a right triangle with altitude OT''. Since the altitude drawn to the hypotenuse of a right triangle is the mean proportion of the segments it cuts, we have

$$\frac{4}{r} = \frac{r}{9} \quad \text{or} \quad r = 6$$

3. (a) : 1st Solution: Construct line CP parallel to EF and intersecting AB at P. Then $\dfrac{AP}{AC} = \dfrac{AF}{AE}$, that is, $\dfrac{AP}{16} = \dfrac{AF}{2AF}$, so $AP = 8$.



Let a, x, y, α, β, δ and θ be as shown in the diagram. The desired ratio $\dfrac{EG}{GF}$ is the same as $\dfrac{y}{x}$ which we now determine. By the law of sines,

$$\frac{a}{\sin\alpha} = \frac{12}{\sin\theta}; \quad \frac{a}{\sin\beta} = \frac{16}{\sin(180° - \theta)} = \frac{16}{\sin\theta}$$

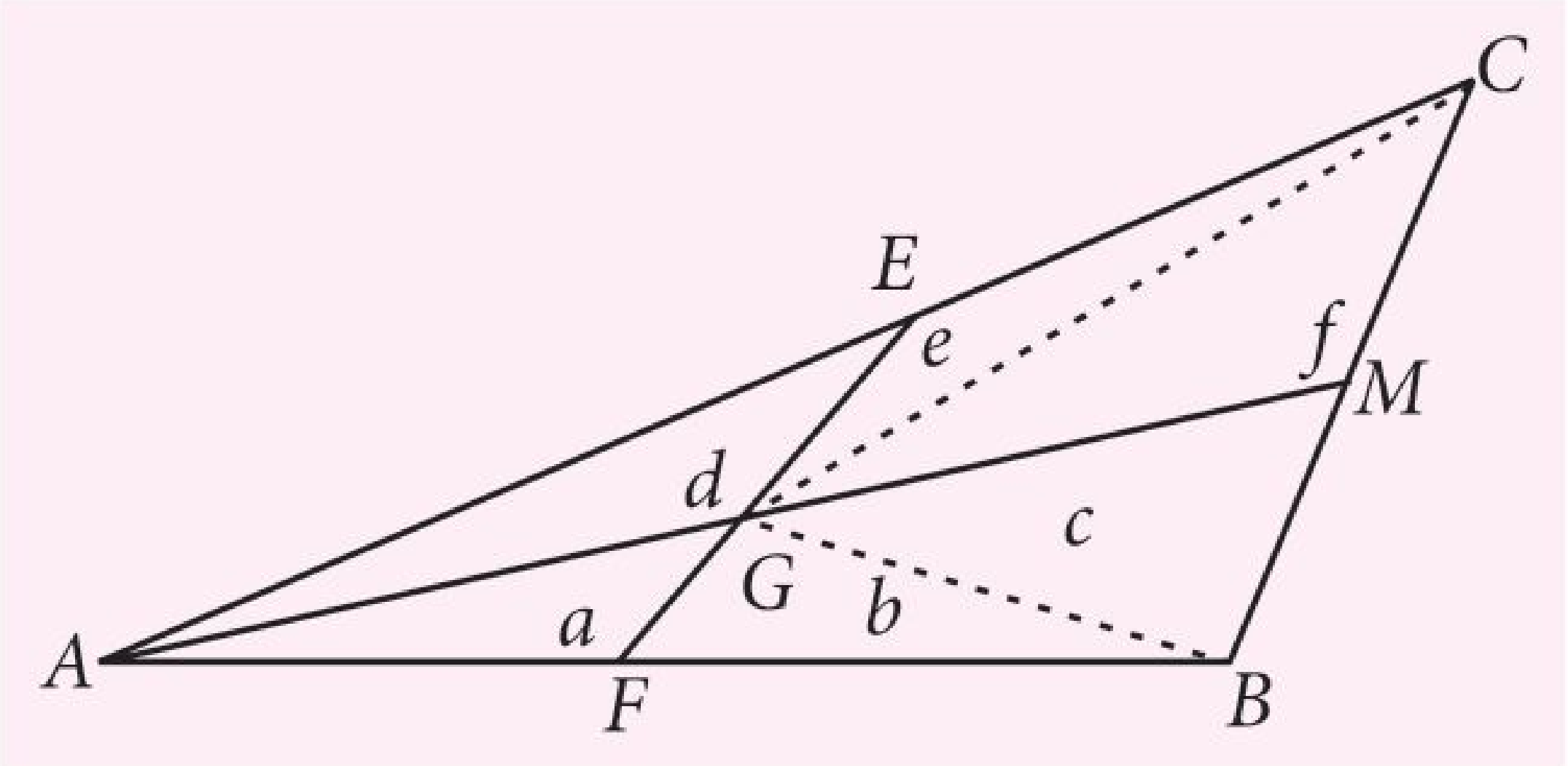
Hence, $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{3}{4}$.

Moreover,

$$\frac{x}{\sin\alpha} = \frac{8}{\sin\delta}; \quad \frac{y}{\sin\beta} = \frac{16}{\sin(180° - \delta)} = \frac{16}{\sin\delta}$$

Hence, $\dfrac{y}{x} = 2\dfrac{\sin\beta}{\sin\alpha} = \dfrac{3}{2}$

2nd Solution :



Join GB and GC. Triangle ABC is subdivided into six smaller triangles whose areas are denoted by a, b, c, d, e, f, as indicated in the diagram. Triangles AEG and AFG have the common vertex A, so their areas are in the ratio EG to GF. Thus $\dfrac{EG}{GF} = \dfrac{d}{a}$, and this we now calculate.

Triangles ACM and ABM have equal areas, so

$$d + e + f = a + b + c$$

Similarly $f = c$, and hence $d + e = a + b$ by subtraction. Let x be the length AF, so that $AE = 2x$, $FB = 12 - x$, $EC = 16 - 2x$. Then

$$\frac{b}{a} = \frac{FB}{FA} = \frac{12 - x}{x} \text{ and } \frac{e}{d} = \frac{EC}{EA} = \frac{16 - 2x}{2x}$$

Hence

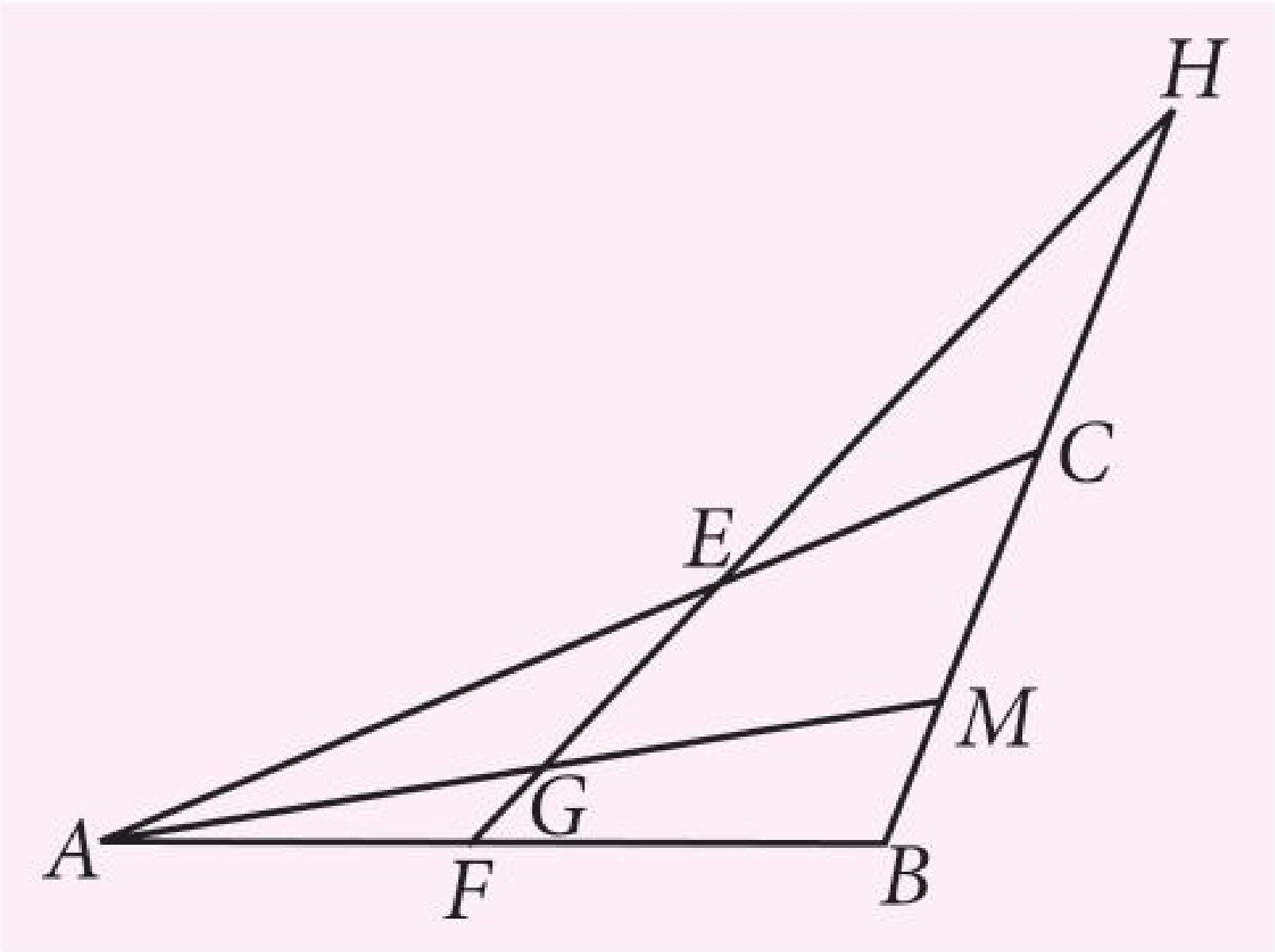
$$a+b=a+\frac{12-x}{x}a=a\frac{12}{x}$$

and $d+e=d+\frac{16-2x}{2x}d=d\frac{16}{2x}$.

Thus $a + b = d + e$ becomes

$$a\frac{12}{x}=d\frac{16}{2x} \quad \text{or} \quad 3a=2d, \ \text{so} \ \frac{d}{a}=\frac{3}{2}$$

3rd Solution :



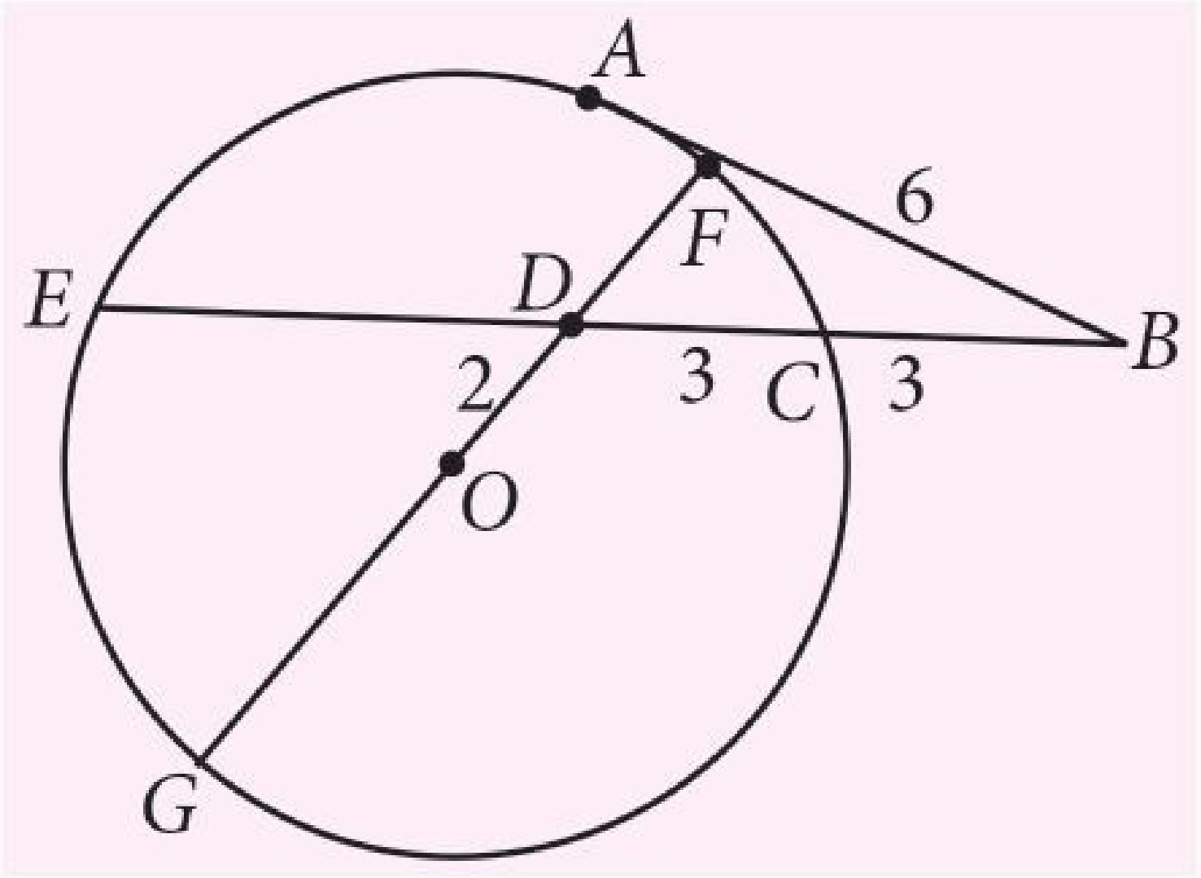
Extend BC and FE until they intersect in a point H; see the figure. The collinear points A, G, M lie on sides (or their extensions) BF, FH, HB of ΔFBH respectively. They also lie on extensions of sides CE, EH, HC of ΔECH. We may therefore apply Menelaus's theorem and find

$$\frac{HG}{FG}\cdot\frac{FA}{BA}\cdot\frac{BM}{HM}=1,\ \frac{HG}{EG}\cdot\frac{EA}{CA}\cdot\frac{CM}{HM}=1$$

Since $CM = BM$ and $EA = 2FA$, division of the first equation by the second yields

$$\frac{EG}{FG}=2\frac{BA}{CA}=2.\frac{12}{16}=\frac{3}{2}$$

4. (e) : In the figure, E is the point of intersection of the circle and the extension of DB, and FG is the diameter passing through D. Let r denote the radius of the circle.



Then, $(BC)(BE) = (AB)^2$

$3(DE + 6) = 36$, $DE = 6$.

Also $(DE)(DC) = (DF)(DG)$,

$6{\cdot}3 = (r - 2)(r + 2)$,

$18 = r^2 - 4$, $r=\sqrt{22}$.

5. (b) : If $MNPQ$ is convex, then A is the sum of areas of the triangles into which $MNPQ$ is divided by diagonal MP, so that

$$A=\frac{1}{2}ab\sin N+\frac{1}{2}cd\sin Q\cdot$$

Similarly, dividing $MNPQ$ with diagonal NQ yields

$$A=\frac{1}{2}ad\sin M+\frac{1}{2}bc\sin P$$

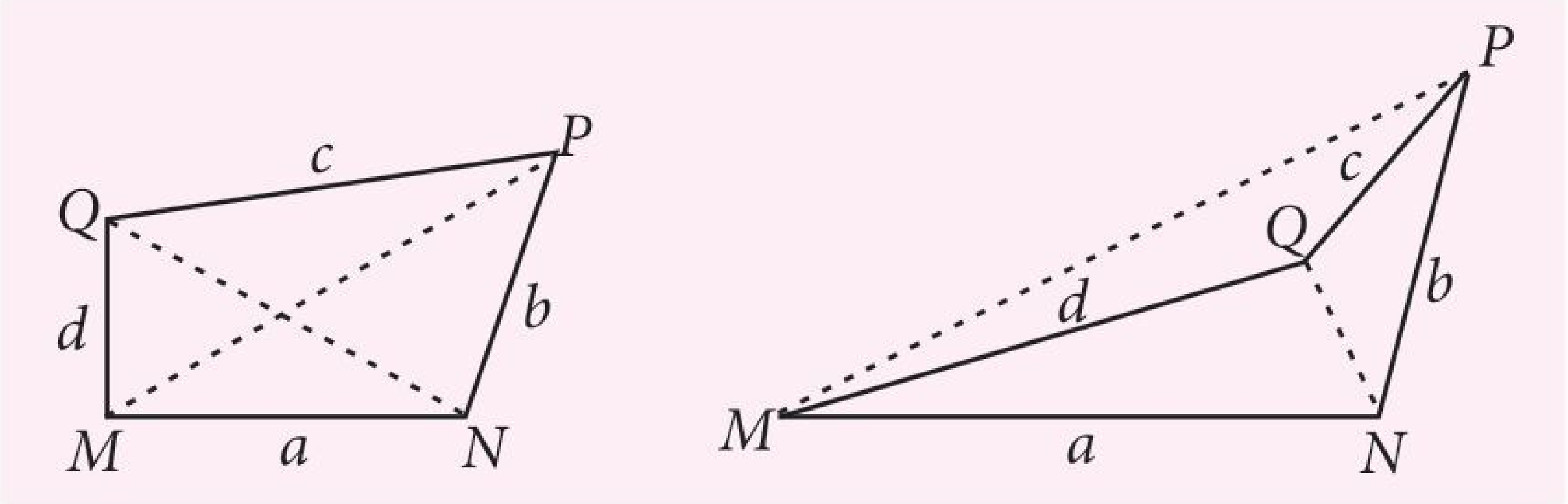
We show below that these two equations for A hold also if $MNPQ$ is not convex. Therefore, in any case,

$$A\le\frac{1}{4}(ab+cd+ad+bc)=\frac{a+c}{2}.\frac{b+d}{2}$$

The inequality is an equality if and only if

$\sin M = \sin N = \sin P = \sin Q = 1$,

i.e. if and only if $MNPQ$ is a rectangle.



If $MNPQ$ is not convex, for example if interior angle Q of quadrilateral $MNPQ$ is greater than 180° as shown in Figure 2, then A is the difference

Area of ΔMNP – area of ΔMQP

so that $A=\frac{1}{2}ab\sin N-\frac{1}{2}cd\sin\angle PQM$

$$=\frac{1}{2}ab\sin N-\frac{1}{2}cd\sin(360°-\angle MQP)$$

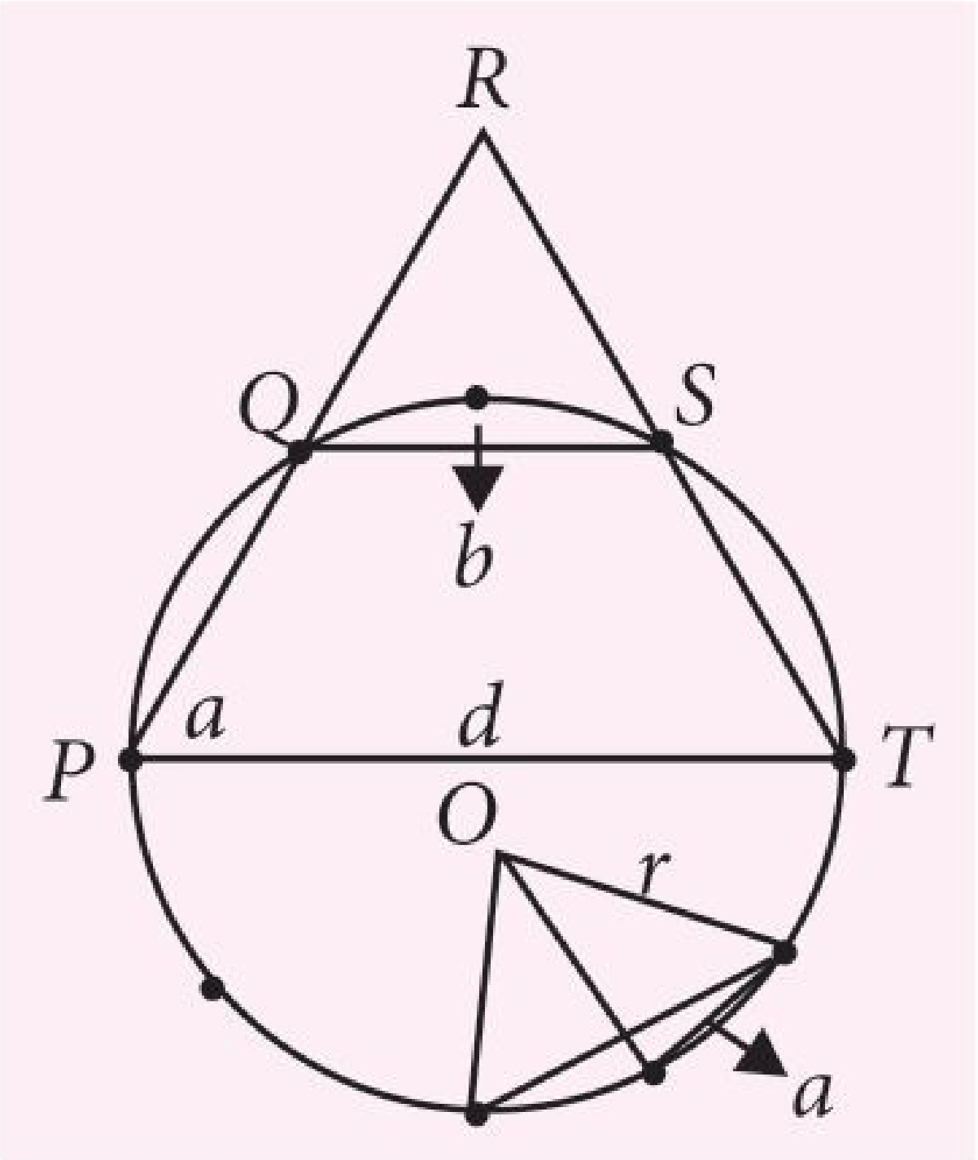
$$=\frac{1}{2}ab\sin N+\frac{1}{2}cd\sin\angle MQP$$

6. (a) : 1st Solution: In the figure, sides PQ and TS of the regular nonagon have extended to meet at R and the circumscribed circle has been drawn. Each side of the nonagon subtends an arc of $\frac{360°}{9}=40°$; therefore

$\angle TPQ = \angle STP =\frac{1}{2}\cdot 3\cdot 40°=60°$.

Since $QS \parallel PT$, it follows that both ΔPRT and ΔQRS are equilateral. Hence

$d = PT = PR = PQ + QR = PQ + QS = a + b$.



2nd Solution: Inscribe the nonagon in a circle of radius r. Since the chords of lengths a, b, d subtend central angles of 40°, 80°, 160° respectively, we have

$a = 2r\sin20°$, $b = 2r\sin40°$, $d = 2r\sin80°$.

By means of the identity

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

with $x = 40°$, $y = 20°$

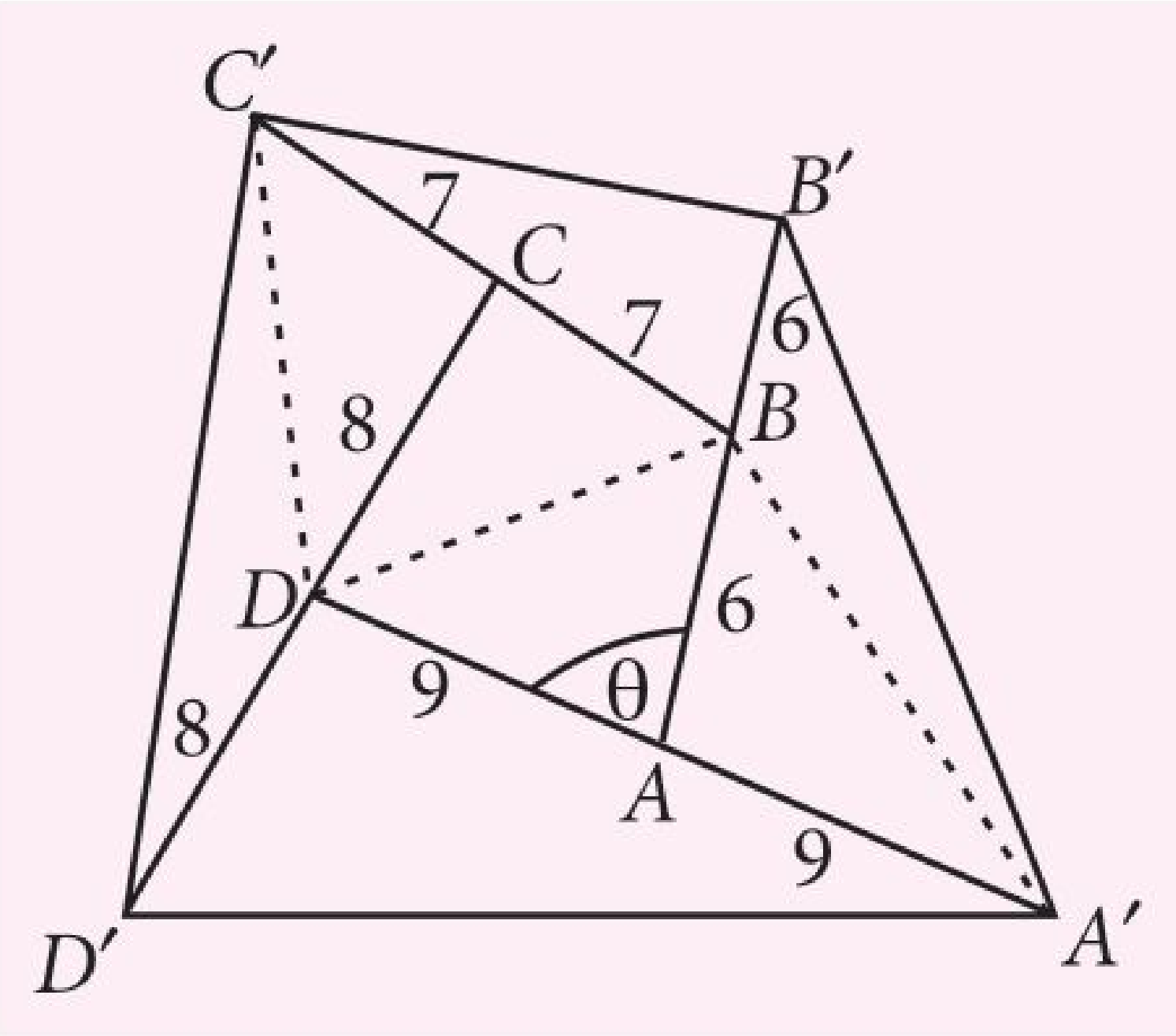
we obtain $\sin 40° + \sin 20° = 2\sin 30° \cos 10°$.

Since $\sin 30° = 1/2$, this yields

$$\sin 40° + \sin 20° = \cos 10° = \sin 80°,$$

and therefore $b + a = d$.

7. (d) :



Since the length of base AA' of $\Delta AA'B$ is the same as the length of base AD of ΔABD, and the corresponding altitude of $\Delta AA'B'$ has twice the length of the corresponding altitude of ΔABD.

Area $\Delta AA'B'$ = 2area ΔADB,

(Alternatively, we could let θ be the measure of $\angle DAB$ and odserve

$$\text{Area }\ \Delta AA'B'=\frac{1}{2}(AD)(2AB)\sin(180°-\theta)$$
$$=2\left[\frac{1}{2}(AD)(AB)\sin\theta\right]=2\text{ area }\Delta ABD)$$

Similarly, area $\Delta BB'C'$ = 2 area ΔBAC,

area $\Delta CC'D'$ = 2 area ΔCBD,

area $\Delta DD'A'$ = 2 area ΔDCA.

Therefore, area $A'B'C'D'$

=(area $\Delta AA'B'$ + area $\Delta BB'C'$) + (area $\Delta CC'D'$ + area $\Delta DD'A'$) + area $ABCD$

= 2(area ΔABD + area ΔBAC) + 2(area ΔCBD + area ΔDCA) + area $ABCD$

= 5 area $ABCD$ = 50.

8. (b) : 1st Solution: Substitute $x = 1$ into the functional equation and solve for the first term on the right side to obtain $f(y + 1) = f(y) + y + 2$.

Since $f(1) = 1$, one sees by successively substituting $y = 2, 3, 4, \ldots$ that $f(y) > 0$ for every positive integer. Therefore, for y a positive integer,

$$f(y + 1) > y + 2 > y + 1,$$

and $f(n) = n$ has no solutions for integers $n > 1$. Solving the above equation for $f(y)$ yields

$$f(y) = f(y + 1) - (y + 2)$$

Successively substituting $y = 0, -1, -2, \ldots$ into this equation yields $f(0) = -1$, $f(-1) = -2$, $f(-2) = -2$, $f(-3) = -1$, $f(-4) = 1$. Now $f(-4) > 0$ and, for $y < -4$, $-(y + 2) > 0$. Thus, for $y < -4$, $f(y) > 0$. Therefore, $f(n) \neq n$ for $n < -4$; and the solutions $n = 1, -2$ are the only ones.

2nd Solution: We write the functional equation in the form

$$f(x + y) - f(y) = f(x) + xy + 1.$$

Setting $x = 1$ and using the given value $f(1) = 1$, we find

$$f(y + 1) - f(y) = y + 2$$

We now set $y = 0, 1, 2, \ldots, n$ successively, then $y = 0, -1, -2, \ldots, -m$ successively, and obtain

$$\begin{array}{ll} f(1) - f(0) = 2 \text{ (so } f(0) = -1) & f(0) - f(-1) = 1 \\ f(2) - f(1) = 3 & f(-1) - f(-2) = 0 \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ f(n) - f(n-1) = n + 1 & f(-(m-1)) - f(-m) \\ & \qquad = -(m-2). \end{array}$$

Adding the set of equations in the left column and cancelling like terms with opposite signs yields

$$f(n)-f(0)=2+3+\ldots+n+1=-1+\sum_{i=1}^{n+1} i$$

Recalling that the sum of the first k positive integers is $\frac{1}{2}k(k+1)$ and using the value $f(0) = -1$, we obtain

$$f(n)+1=\frac{(n+1)(n+2)}{2}-1$$

(1). $f(n)=\frac{1}{2}(n^2+3n-2)$ for each non-negative integer n.

The same procedure applied to the column of equations on the right above shows that equation (1) is valid also for negative integers; thus (1) holds for all integers.

The equation $f(n) = n$ is equivalent to

$$n^2 + 3n - 2 = 2n,$$
$$n^2 + n - 2 = (n + 2)(n - 1) = 0,$$
$$n = 1 \quad\text{or}\quad n = -2;$$

So there is only one integer solution other than $n = 1$.

Remark: The computations involving the equations in the right column above can be avoided by setting $y = -x$ in the original equation to obtain

$$f(x) + f(-x) = f(0) + x^2 - 1,$$

or since $f(0) = -1$,

(2). $f(x) = -f(-x) + x^2 - 2$ for all x.

Let x be negative and substitute (1) in (2) to get

$$f(x)=-\frac{1}{2}\left[(-x)^2+3(-x)-2\right]+x^2-2=\frac{1}{2}(x^2+3x-2)$$

Thus negative integers as well as positive satisfy (1). In fact, it is easy to check that the function

$$f(x)=\frac{x^2+3x-2}{2}$$

satisfies the given functional equation for all real x and y.

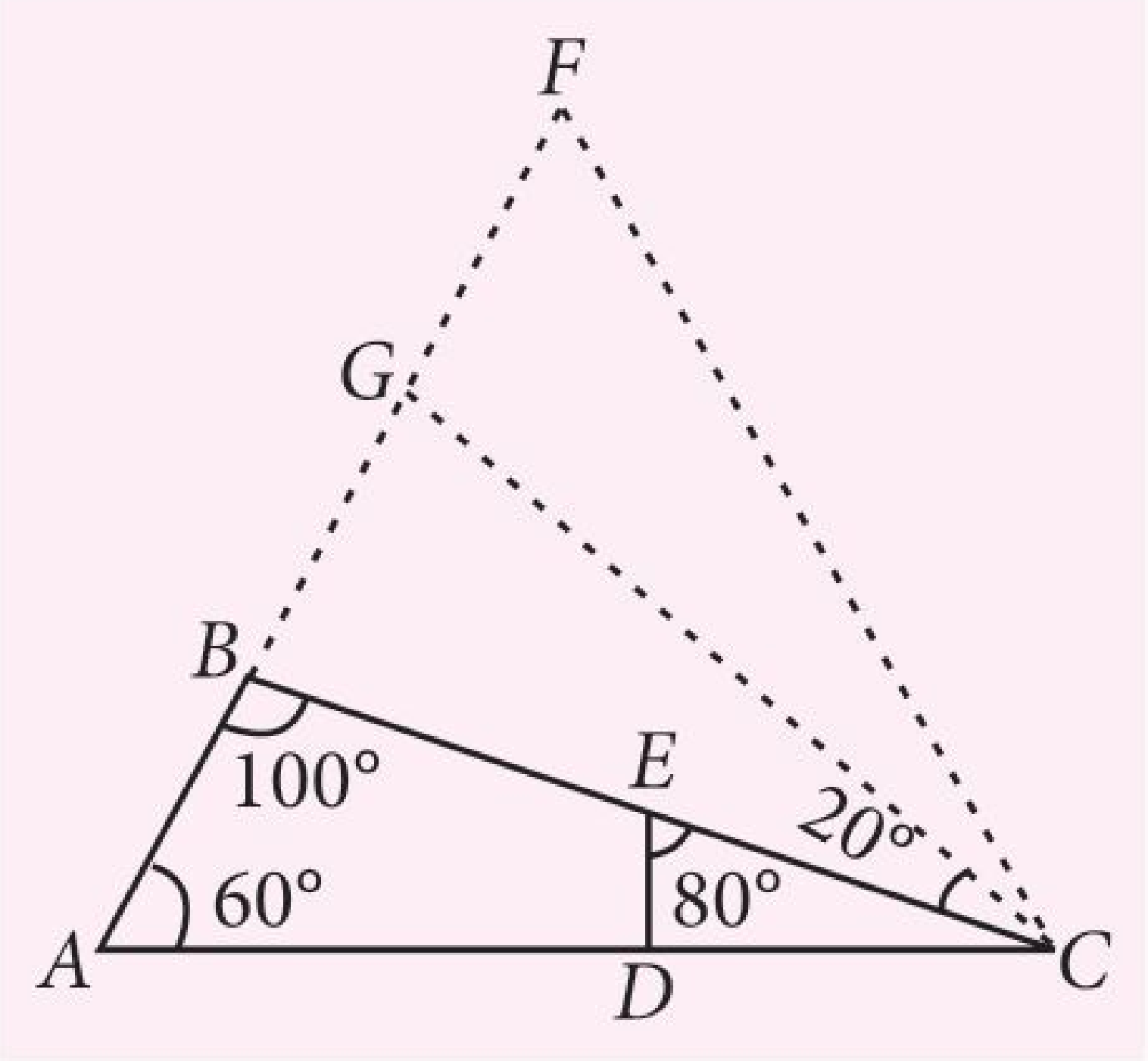
Remark: The technique used in the second solution is frequently applied to solve many commonly occurring functional equations involving the expression $\Delta f(x) = f(x + 1) - f(x)$; $\Delta f(x)$ is called the "first difference" of $f(x)$ and behaves, in many ways, like the first derivative $f'(x)$ of $f(x)$. For example

$$f'(x)=x+2 \text{ implies } f(x)=\frac{x^2}{2}+2x+c$$

$$\Delta f(x)=x+2 \text{ implies } f(x)=\frac{x(x-1)}{2}+2x+c$$

where c is a constant. The study of functions by means of $\Delta f(x)$ is called the theory of finite differences.

9. (b) : Let F be the point on the extension of side AB past B for which $AF = 1$. Since $AF = AC$ and $\angle FAC = 60°$, ΔACF is equilateral. Let G be the point on line segment BF for which $\angle BCG = 20°$. Then ΔBCG is similar to ΔDCE and $BC = 2(EC)$. Also ΔFGC is congruent to ΔABC.



Therefore, area ΔACF = (area ΔABC + area ΔGCF) + area ΔBCG,

$$\frac{\sqrt{3}}{4}=2\text{ area }(\Delta ABC)+4\text{ area }(\Delta CDE)$$

$$\frac{\sqrt{3}}{8}=\text{area }(\Delta ABC)+2\text{ area }(\Delta CDE)$$

10. (d) : $\log_b \sin x = a$; $\sin x = b^a$; $\sin^2 x = b^{2a}$;

$$\cos x=(1-b^{2a})^{1/2};\ \log_b\cos x=\frac{1}{2}\log_b(1-b^{2a})$$

11. (c) : The given information implies that $a_n > a_{n-1}$ if and only if $n + c$ is a perfect square. Since $a_2 > a_1$ and $a_5 > a_4$, it follows that $2 + c$ and $5 + c$ are both squares. The only squares differing by 3 are 1 and 4; here $2 + c = 1$, so $c = -1$.

Now $a_2=3=b[\sqrt{2+c}]+d=b[\sqrt{1}]+d=b+d$.

Hence, $b + c + d = 3 - 1 = 2$. (Although we only need to find $b + d$ here, it is easy to see by setting $n = 1$ that $d = 1$, and hence $b = 2$.)

Remark: The last member of the k^{th} string of equal terms occupies the position $1 + 3 + 5 +...+ (2k - 1) = k^2$ in the sequence. Its successor is $a_{k^2+1}=a_{k^2}+2$.

Therefore, $[\sqrt{k^2+1+c}]-[\sqrt{k^2+c}]=\frac{2}{b}=1$

so $c = -1$ and $b = 2$.

12. (a) : 1st Solution: If the last equation is multipled by 3 and added to the first equation, we obtain

$$4x^2 + 2y^2 + 23z^2 = 331.$$

Clearly, z^2 is odd and less than 25, so $z^2 = 1$ or 9. This leads to the two equations

$$2x^2 + y^2 = 154 \quad\text{and}\quad 2x^2 + y^2 = 62$$

Both of which are quickly found to have no solutions. Note that we made no use here of the second of the original equations.

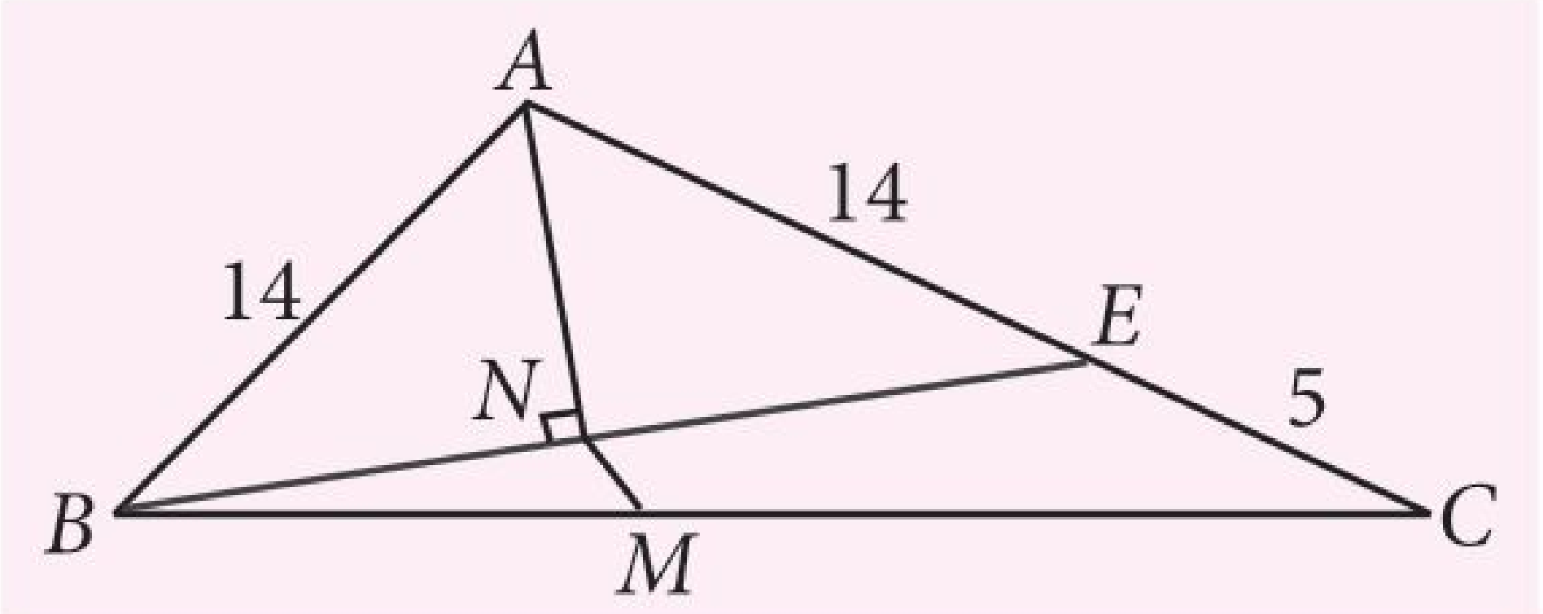
2nd Solution: Adding the given equations and rearranging the terms of the resulting equation yields

$$(x^2 - 2xy + y^2) + (y^2 + 6yz + 9z^2) = 175$$

or $\quad (x - y)^2 + (y + 3z)^2 = 175$

The square of an even integer is divisible by 4; the square of an odd integer, $(2n + 1)^2$, has remainder 1 when divided by 4. So the sum of two perfect squares can only have 0, 1 or 2 as a remainder when divided by 4. But 175 has remainder 3 upon division by 4, and hence the left and right sides of the equation above cannot be equal. Thus there are no integral solutions.

13. (b) : In the adjoining figure, BN is extended past N and meets AC at E. Triangle BNA is congruent to ΔENA. Since $\angle BAN = \angle EAN$, $AN = AN$ and $\angle ANB = \angle ANE$. Therefore N is the midpoint of BE, and $AB = AE = 14$. Thus $EC = 5$. Since MN is the line joining the midpoints of sides BC and BE of ΔCBE, its length is $\frac{1}{2}(EC) = \frac{5}{2}$.



14. (d) : The probability that the first 6 is tossed on the k^{th} toss is the product

$$\left(\begin{array}{c}\text{probability that never a 6 was}\\ \text{tossed in the previous }(k\text{-}1)\text{ tosses}\end{array}\right)\left(\begin{array}{c}\text{probability that a 6 is}\\ \text{tossed on the }k^{\text{th}}\text{ toss}\end{array}\right)$$

$$=\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right).$$

The probability that Carol will toss the first 6 is the sum of the probabilities that she will toss the first 6 on her first turn (3rd toss of the game), on her second turn (6th toss of the game), on her third turn, etc. This sum is

$$\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}+\ldots+\left(\frac{5}{6}\right)^{3n-1}\cdot\frac{1}{6}+\ldots,$$

an infinite geometric series with first term $a=\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}$

and common ratio $r=\left(\frac{5}{6}\right)^3$. This sum is

$$\frac{a}{1-r}=\frac{\frac{5^2}{6^3}}{1-\left(\frac{5^3}{6^3}\right)}=\frac{5^2}{6^3-5^3}=\frac{25}{91}$$

15. (d) : Let $d_1=a+\sqrt{b}$ and $d_2=a-\sqrt{b}$, where $a = 15$ and $b = 220$. Then using the binomial theorem, we may obtain

$$d_1^n+d_2^n=2\left[a^n+\binom{n}{2}a^{n-2}b+\binom{n}{4}a^{n-4}b^2+\ldots\right],$$

where n is any positive integer. Since fractional powers of b have been eliminated in this way, and since a and b are both divisible by 5, we may conclude that $d_1^n+d_2^n$ is divisible by 10.

We now apply the above result twice, taking $n = 19$ and $n = 82$. In this way we obtain

$$d_1^{19}+d_2^{19}=10k_1 \text{ and } d_1^{82}+d_2^{82}=10k_2,$$

k_1 and k_2 are positive integers. Adding and rearranging these results gives

$$d_1^{19}+d_2^{19}=10k_1 \text{ and } d_1^{82}+d_2^{82}=10k_2,$$

where $k=k_1+k_2$. But $d_2=15-\sqrt{220}=\frac{5}{15+\sqrt{220}}<\frac{1}{3}$.

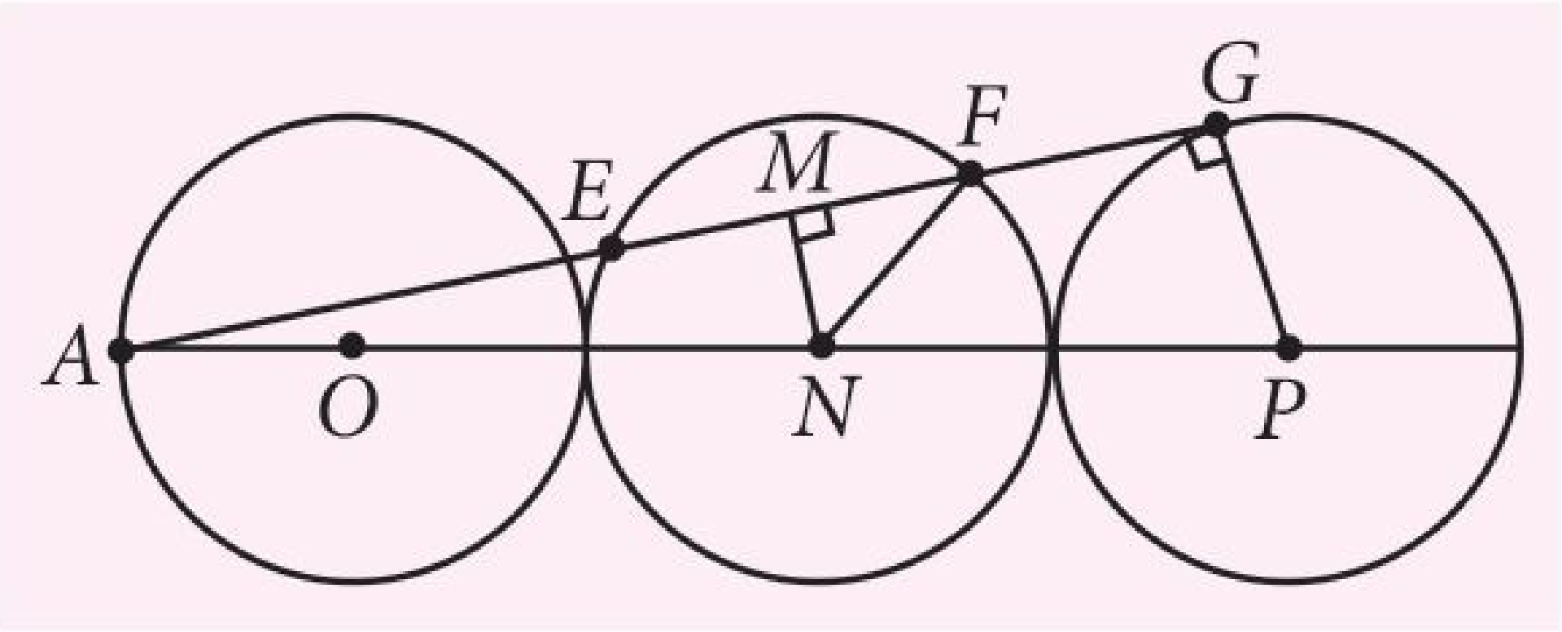
Therefore, $d_2^{19}+d_2^{82}<1$. It follows that the units digit of $10k-(d_2^{19}+d_2^{82})$ is 9.

16. (c) : In the adjoining figure, MN is perpendicular to AG at M, and NF and PG are radii.

Since $\Delta AMN \sim \Delta AGP$, it follows that

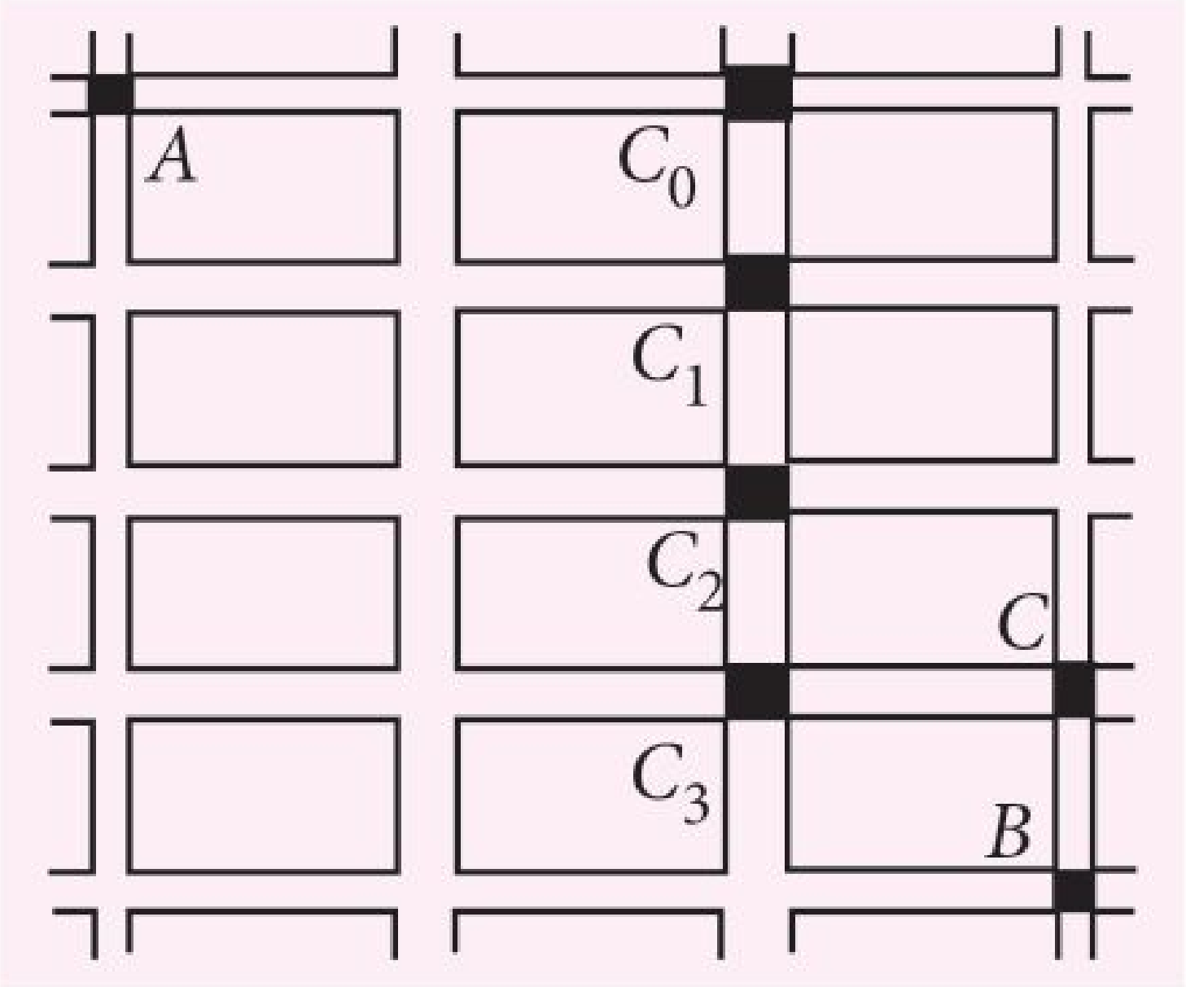
$$\frac{MN}{AN}=\frac{GP}{AP} \quad\text{or,}\quad \frac{MN}{45}=\frac{15}{75}$$

Thus $MN = 9$. Applying the Pythagorean theorem to triangle MNF yields $(MF)^2 = (15)^2 - 9^2 = 144$, so $MF = 12$. Therefore $EF = 24$.



17. (d) : 1st Solution: The probability that the student passes through C is the sum from $i = 0$ to 3 of the probabilities that he enters intersection C_i in the adjoining figure and goes east. The number of paths from A to C_i is $\binom{2+i}{2}$, because each such path has 2 eastward block segments and they can occur in any order. The probability of taking any one of these paths to C_i and then going east is $\left(\frac{1}{2}\right)^{3+i}$ because there are $3 + i$ intersections along the way (including A and C_i) where an independent choice with probability 1/2 is made. So the answer is

$$\sum_{i=0}^{3}\binom{2+i}{2}\left(\frac{1}{2}\right)^{3+i}=\frac{1}{8}+\frac{3}{16}+\frac{6}{32}+\frac{10}{64}=\frac{21}{32}.$$



2nd Solution: One may construct a tree-diagram of the respective probabilities, obtaining the values step-by-step as shown in the scheme to the right (the final 1 also serves as a check on the computations).

$$\begin{array}{ccccccc}
1 & \rightarrow & \frac{1}{2} & \rightarrow & \frac{1}{4} & \rightarrow & \frac{1}{8} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{2} & \rightarrow & \frac{1}{2} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{5}{16} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{4} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{3}{8} & \rightarrow & \frac{1}{2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{8} & \rightarrow & \frac{1}{4} & \rightarrow & \frac{5}{16} & \rightarrow & \frac{21}{32} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{16} & \rightarrow & \frac{3}{16} & \rightarrow & \frac{11}{32} & \rightarrow & 1
\end{array}$$

It is important to recognize that not all twenty of the thirty-five paths leading from A to B through C are equally likely; hence answer (c) is incorrect.

18. (c) : We recall the theorem that complex roots of polynomials with real coefficients come in conjugate pairs. Though not applicable to the given polynomial, that theorem is proved by a technique which we can use to work this problem too. Namely, conjugate both sides of the original equation

$$0 = c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0,$$

obtaining

$$0 = c_4\bar{z}^4 - ic_3\bar{z}^3 + c_2\bar{z}^2 - ic_1\bar{z} + c_0$$

$$= c_4(-\bar{z})^4 + ic_3(-\bar{z})^3 + c_2(-\bar{z})^2 - ic_1(-\bar{z}) + c_0$$

That is, $-\bar{z} = -a + ib$ is also a solution of the original equation. (One may check by example that neither $-a - bi$ nor $a - bi$ nor $b + ai$ need be a solution.) For instance, consider the equation $z^4 - iz^3 = 0$ and the solution $z = i$. Here $a = 0$, $b = 1$. Neither $-i$ nor 1 is a solution. [Alternatively, the substitution $z = i\omega$ into the given equation makes the coefficients real and the above quoted theorem applicable.

PART B

1. Suppose $(x_1, x_2, ..., x_n)$ is a rearrangement of $(1, 2, ..., n)$ such that $|x_1 - 1|, |x_2 - 1|, ..., |x_n - 1|$ are all distinct. Let us show that $n \equiv 0$ or $1 \pmod 4$. Now

$$\sum_{i=1}^{n}(x_i - i) = 0 \qquad ...(1)$$

Since $|x_1 - 1|, |x_2 - 2|, ..., |x_n - n|$ are all distinct, they have to be a rearrangement of $0, 1, ..., n - 1$; therefore,

$$\sum_{i=1}^{n}|x_i - i| = \frac{n(n-1)}{2} \qquad ...(2)$$

Since for any $k \in z|k| \equiv k \pmod 2$,

$$\sum_{i=1}^{n}|x_i - i| \equiv \sum_{i=1}^{n}(x_i - i) \pmod 2$$

Hence, it follows from (1) and (2) that $\frac{n(n-1)}{2}$ is even.

Therefore $4|n(n - 1)$; *i.e.*, 4 divides either n or $n - 1$.

2. 1st Solution: Since $p(n)p(n + 1)$ is of fourth degree n with leading coefficient 1, m, if it exists, must be of the form $n^2 + tn + s$

Therefore $p(0)\ p(1) = p(s)$

$\Rightarrow\ b^2 + ab + b = s^2 + as + b \Rightarrow s^2 - b^2 = a(b - s)$

$\Rightarrow\ (s - b)(s + b) = a(b - s) \Rightarrow (s - b)(s + a + b) = 0$

Thus we have $s = b$ or $s = -(a + b)$. Taking $s = b$, we have,

$$(n^2 + an + b)((n + 1)^2 + a(n + 1) + b)$$
$$= (n^2 + tn + b)^2 + a(n^2 + tn + b) + b$$

Equating the coefficients of n^3, we obtain $t = a + 1$. Taking $m = n^2 + (a + 1)n + b$, it can be routinely verified that $p(n)p(n + 1) = p(m)$ for all $n \in N$.

2nd Solution: Writing $p(x) = (x + \alpha)(x + \beta)$, we have $\alpha + \beta = a$ and $\alpha\beta = b$. Now

$(n + \alpha)(n + 1 + \beta) = n^2 + (\alpha + \beta + 1)n + \alpha\beta + \alpha$

$= n^2 + (a + 1)n + b + \alpha$

Similarly $(n + \beta)(n + 1 + \alpha) = m + \beta$

where $m = n^2 + (a + 1)n + b$. Therefore

$$p(n)p(n + 1) = (n + \alpha)(n + \beta)\ (n + 1 + \alpha)(n + 1 + \beta)$$
$$= (m + \alpha)(m + \beta) = p(m).$$

3. 1st Solution: Choose M as the origin. Let A, B, C, D be $(2a, 0), (0, 2b), (2c, 0), (0, 2d)$ respectively. Then

$$AM{\cdot}MC = BM{\cdot}MD \Rightarrow ac = bd$$

(in magnitude as well as in sign).

O being the intersection of the perpendicular bisectors of AC and BD, is $(a + c, b + d)$. Since K, L are (a, b), (c, d) respectively, the midpoint of KL is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ which is also the mid-point of OM. Therefore $OKML$ is a parallelogram.

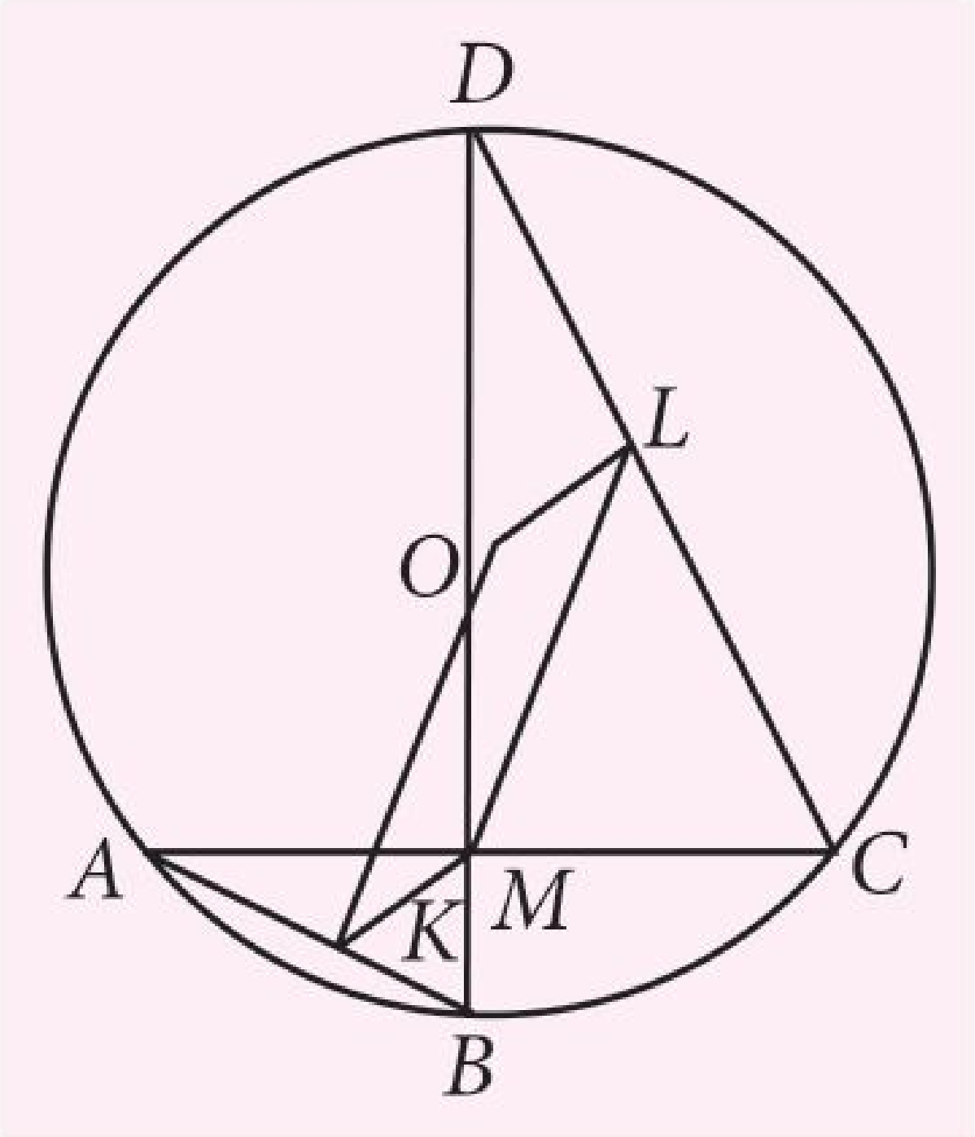
2nd Solution: Let R be the radius of the circle. Then in ΔADB, $AB = 2R\sin\angle ADB$.

Also in ΔCAD, $CD = 2R\sin\angle CAD = 2R\cos\angle ADB$

($\because$ In ΔADM, $\angle ADB + \angle CAD = 90°$). Therefore

$$AB^2 + CD^2 = 4R^2.$$

Now, $AK^2 + OK^2 = AO^2 \Rightarrow \frac{1}{2}AB^2 + OK^2 = R^2$



and therefore $OK = \frac{1}{2}CD = ML$ ($\because$ In the right angled triangle CMD, L is the mid-point of the hypotenuse).

Similarly $OL = KM$.

3rd Solution: Let $\angle KAM = \theta$.

In ΔAKM, $AK = KM \Rightarrow \angle AMK = \theta$

$\therefore\ \angle AMK = 180° - 2\theta.$

So. $\angle OKM = 180° - 2\theta - 90° = 90° - 2\theta$
By similar argument, since
$\angle MDC = \theta$, we get $\angle OLM = 90° - 2\theta$. Further,
$$\angle KML = \angle KMA + \angle AMD + \angle DML = \theta + 90° + \theta = 90° + 2\theta$$
Thus, three angles $\angle OKM$, $\angle OLM$, $\angle KML$ of $OKML$ are $90° - 2\theta$, $90° - 2\theta$, $90° + 2\theta$ respectively; so the remaining is $90° + 2\theta$ and hence $OKML$ is a parallelogram.

4. Let $(a_i)_{i=1}^{16}$ be one such sequence. For any $k \le 9$, by using (b), we have
$$a_k + a_{k+1} + \dots + a_{k+6} = -1 = a_{k+1} + \dots + a_{k+7}$$
$$\therefore \quad a_k = a_{k+7}.$$
Similarly, by using (c), we have $a_k = a_{k+11}$, for all $k \le 5$. Thus by (a), (b) and (c), we have the following data. For all $i, j \le 16$.
If $i + j = 17$ or $|i - j| = 7$ or 11, then $a_i = a_j$.
Using this, we have
$$a_1 = a_8 = a_{15} = a_4 = a_{11} = a_6 = a_{13} = a_2 = a_9 = a_{16} = a_5 = a_{12}$$
and $a_3 = a_{10} = a_7 = a_{14}$
Now using (b) and (c) for the first 7 and 11 terms respectively, we get $5a_1 + 2a_3 = -1$ and $8a_1 + 3a_3 = 1$; solving, we have $a_1 = 5$ and $a_3 = -13$. Thus the sequence which satisfies our requirement is unique, *viz*,
5, 5, – 13, 5, 5, 5, – 13, 5, 5, –13, 5, 5, 5, –13, 5, 5.

5. For $i = 1, 2, \dots, 6$. Let A_i(resp. B_i) be the set of days on which i^{th} friend is present (resp. absent) at dinner given that
$$|A_i| = |B_i| = 7, \ |A_i \cap A_j| = 5$$
$$|A_i \cap A_j \cap A_k| = 4, \ \ |A_i \cap A_j \cap A_k \cap A_l| = 3$$
$$|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 2$$
and $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 1$
Where i, j, k, l, m vary from 1 to 6 and are distinct. Number of dinners at which at least one friend was present
$$= |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6|$$
$$= \Sigma|A_1| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k|$$
$$\Sigma|A_i \cap A_j \cap A_k \cap A_l| + \Sigma|A_i \cap A_j \cap A_k \cap A_l \cap A_m|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$
$$= \binom{6}{1}7 - \binom{6}{2}5 + \binom{6}{3}4 - \binom{6}{4}3 + \binom{6}{5}2 - \binom{6}{6}1 = 13$$
The total number of dinners is $|A_1| + |B_1| = 14$. Hence the number of dinners I had alone is $14 - 13 = 1$.

6. Let the strength of the class be 100. Let T, E, M, S be the sets of students who passed in Tamil, English, Mathematics and Science, respectively. Then the number of students who failed in all subjects is
$$100 - |T \cup E \cup M \cup S| \ge 100 - (|T| + |E| + |M| + |S|)$$
$$= 100 - (30 + 25 + 20 + 15) = 10$$
Note that equality holds when
$$|T \cup E \cup M \cup S| = |T| + |E| + |M| + |S|;$$
i.e., when these sets are pairwise disjoint.

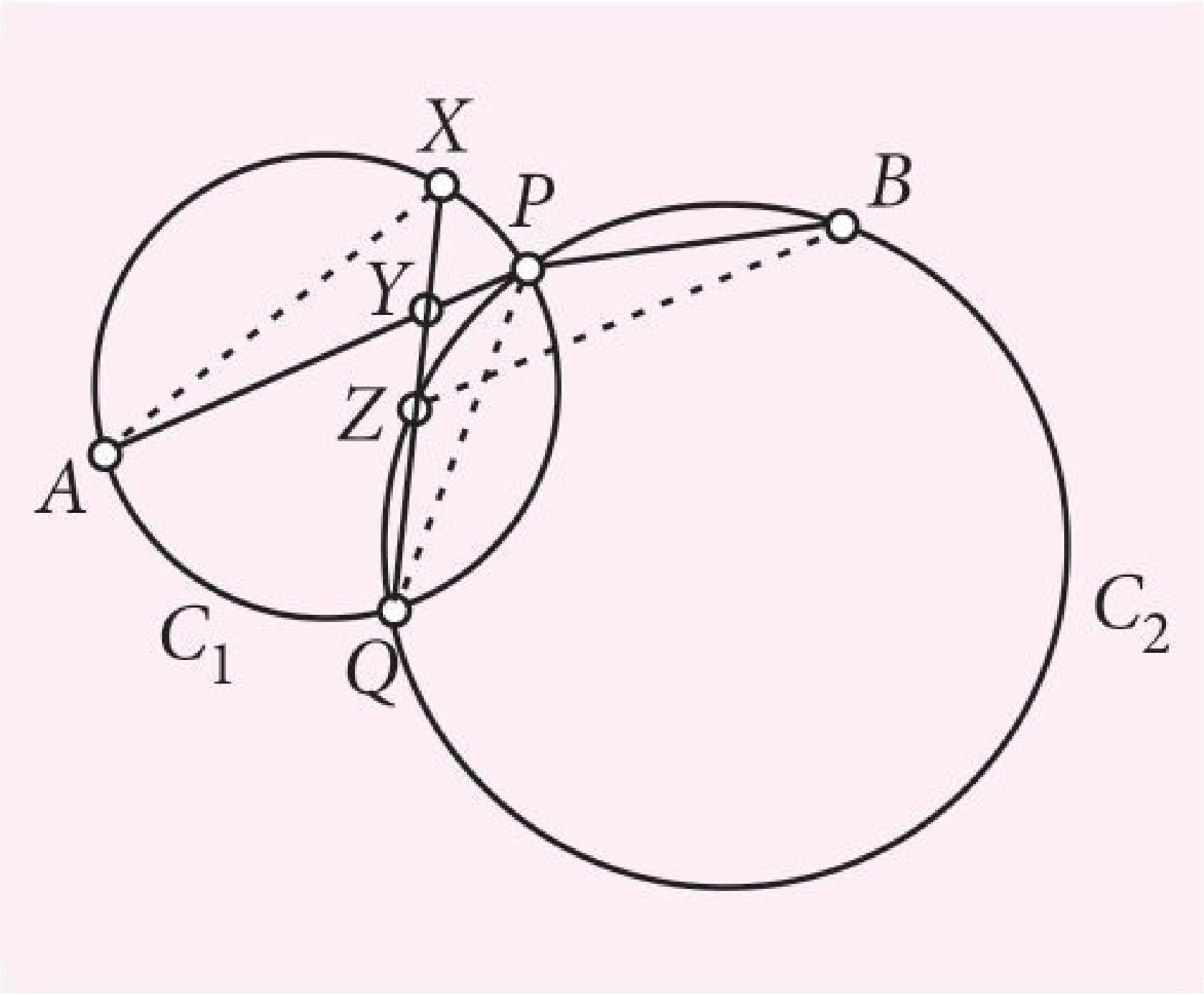
7. Let $[\sqrt{n}] = k$. Then $k^2 < n < (k + 1)^2$. Also since k^3 divides n^2, we have that k^2 divides n^2 and hence k divides n. Thus the only possibilities for n are $n = k^2 + k$ and $n = k^2 + 2k$.

(i) Let $n = k^2 + k$. Then
$$k^3 | n^2 \Rightarrow k^3 | (k^2 + k)^2 = k^4 + 2k^3 + k^2$$
$$\Rightarrow k^3 | k^2 \Rightarrow k = 1 \ \textit{i.e.}, n = 2.$$
(ii) Let $n = k^2 + 2k$. Then
$$k^3 | n^2 \Rightarrow k^3 | (k^2 + 2k)^2 = k^4 + 4k^3 + 4k^2$$
which implies that $k^3 | 4k^2$ or $k | 4$. Therefore, $k = 1, 2$ or 4. When $k = 1, 2, 4$, we get the corresponding values 3, 8 and 24 for n. Thus $n = 2, 3, 8$ and 24 are all positive integers satisfying the given conditions.

8. Eliminating z from the given set of equations, we get
$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$
This factors to
$$(x + y)(x^2 - xy + y^2 + x + y - 2) = 0$$
Case-I: Suppose $x + y = 0$. Then $z = 1$ and $(x, y, z) = (m, -m, 1)$, where m is an integer give one family of solutions.
Case-II : Suppose $x + y \ne 0$. Then we must have
$$x^2 - xy + y^2 + x + y - 2 = 0.$$
This can be written in the form
$$(2x - y + 1)^2 + 3(y + 1)^2 = 12$$
Here there are two possibilities:
$$2x - y + 1 = 0, \ y + 1 = \pm 2;$$
or $2x - y + 1 = \pm 3, \ y + 1 = \pm 1$
Analysing all these cases, we get the following solutions:
$$(0, 1, 0), (-2, -3, 6), (1, 0, 0),$$
$$(0, -2, 3), (-2, 0, 3), (-3, -2, 6).$$

9. **1st Solution:** Join PQ, BZ and AX.

In circle C_2, we have $\angle ZBP = \angle ZQP$; and in circle C_1, we have $\angle PQX = \angle PAX$. Thus, we obtain $\angle ZBA = \angle BAX$. (So BZ is parallel to AX.) The triangles AXY and BZY are then congruent, because by hypothesis $AY = YB$ and angles AYX and YAX are respectively equal to BYZ and YBZ. This congruence gives us $XY = ZY$, which is what we want.



2nd Solution: We use a standard property of intersecting chords of a circle. If AB and CD are two chords of a circle intersecting at O either internally or externally, then $AO \cdot OB = CO \cdot OD$.

In the figure, AP and XQ are two chords of the circle C_1, intersecting externally at Y. So

$$AY \cdot YP = XY \cdot YQ \qquad \text{.... (1)}$$

Similarly, BP and QZ are two chords of the circle C_2 intersecting externally at Y. So

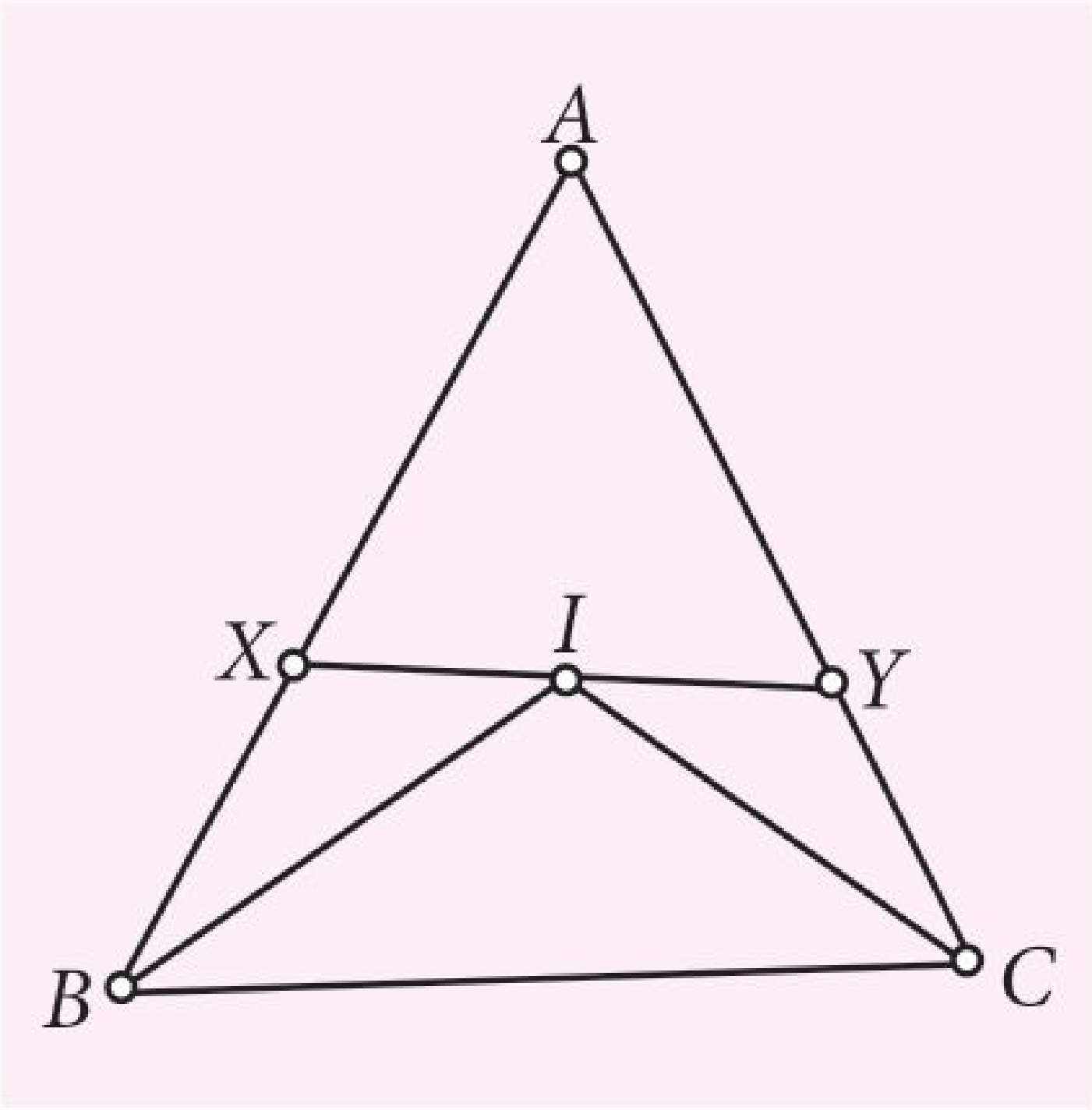
$$YP \cdot YB = YZ \cdot YQ. \qquad \text{.... (2)}$$

The left hand side expressions of (1) and (2) are equal because it is given that $AY = YB$. Therefore the right-hand expressions are equal. This give, on cancelling the factor YQ, the desired relation $XY = YZ$.

10. From the relation $BI^2 = BX \cdot BA$, we see that BI is a tangent to the circle passing through A, X, I at I. Hence

$$\angle BIX = \angle BAI = \frac{A}{2} \qquad \text{.... (1)}$$

[Alternatively, one observes that in triangles BIX and BAI, $\angle IBX$ is common and $\frac{BI}{BX} = \frac{BA}{BI}$. Consequently the two triangles are similar, implying (1).]



Similarly, from the relation $CI^2 = CY \cdot CA$, we obtain

$$\angle CIY = \angle CAI = \frac{A}{2} \qquad \text{... (2)}$$

It is known that

$$\angle BIC = 90° + \frac{A}{2} \qquad \text{.... (3)}$$

From (1), (2), (3) and the fact that X, I, Y are collinear, we obtain

$$\frac{A}{2} + \frac{A}{2} + \left(90° + \frac{A}{2}\right) = 180°.$$

Solving, we get $A = 60°$.

11. **(i)** Since A contains $(n + 1)$ elements of the set $\{1, 2, 3, ..., 2n\}$ some two of the $n + 1$ element must be consecutive. But then any two consecutive integers are relatively prime and we have the desired conclusion.

(ii) We give a proof by making use of the pigeon-hole principle. Write each of the $n + 1$ numbers in the form $2^p \cdot q$, where q is an odd number and p a non negative integer. What are the possible values of q ? Since the numbers of A come from the set $\{1, 2, 3, ... 2n\}$, we see that q can be any one of the n odd numbers 1, 3, 5, 7, ... $(2n - 1)$. As there are $n + 1$ numbers in A, there are $n + 1$ values of q. Hence by the afore-mentioned principle, for some two numbers $a = 2^{p1} \cdot q_1$ and $b = 2^{p2} \cdot q_2$, we must have $q_1 = q_2$. Since $a \neq b$, p_1 is either greater than p_2 or less than p_2. In the former case b divides a, while in the latter case a divides b.

Remark : Strangely, this problem can be solved by induction also.

12. Let S denote the set of all the 25 students in the class, X the set of swimmers in S, Y the set of all weight-lifters and Z the set of all cyclists. Since students in $X \cup Y \cup Z$ all get grades B and C and six students get grades D or E, the number of students in $X \cup Y \cup Z$ $\leq 25 - 6 = 19$. Now assign one point to each of the 17 cyclists, 13 swimmers and 8 weight-lifters. Thus a total of 38 points would be assigned among the students in $X \cup Y \cup Z$. Note that no student can have $|X \cup Y \cup Z| \geq 19$ as otherwise 38 points cannot be accounted for. (For example if there were only 18 student in $(X \cup Y \cup Z)$ the maximum number of points that could be assigned to them is 36). Therefore $|X \cup Y \cup Z| = 19$ and each students $X \cup Y \cup Z$ is in exactly 2 of the sets X, Y, Z. Hence the number of student getting grade $A = 25 - 19 - 6 = 0$, *i.e.* no student gets A grade. Since there are $19 - 8 = 11$ students who are not weight-lifters all these 11 students must be both swimmers and cyclists. (Similarly there are 2 who are both swimmers and weight-lifters and 6 who are both cyclists and weight-lifters).

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YOU ASK
WE ANSWER

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Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

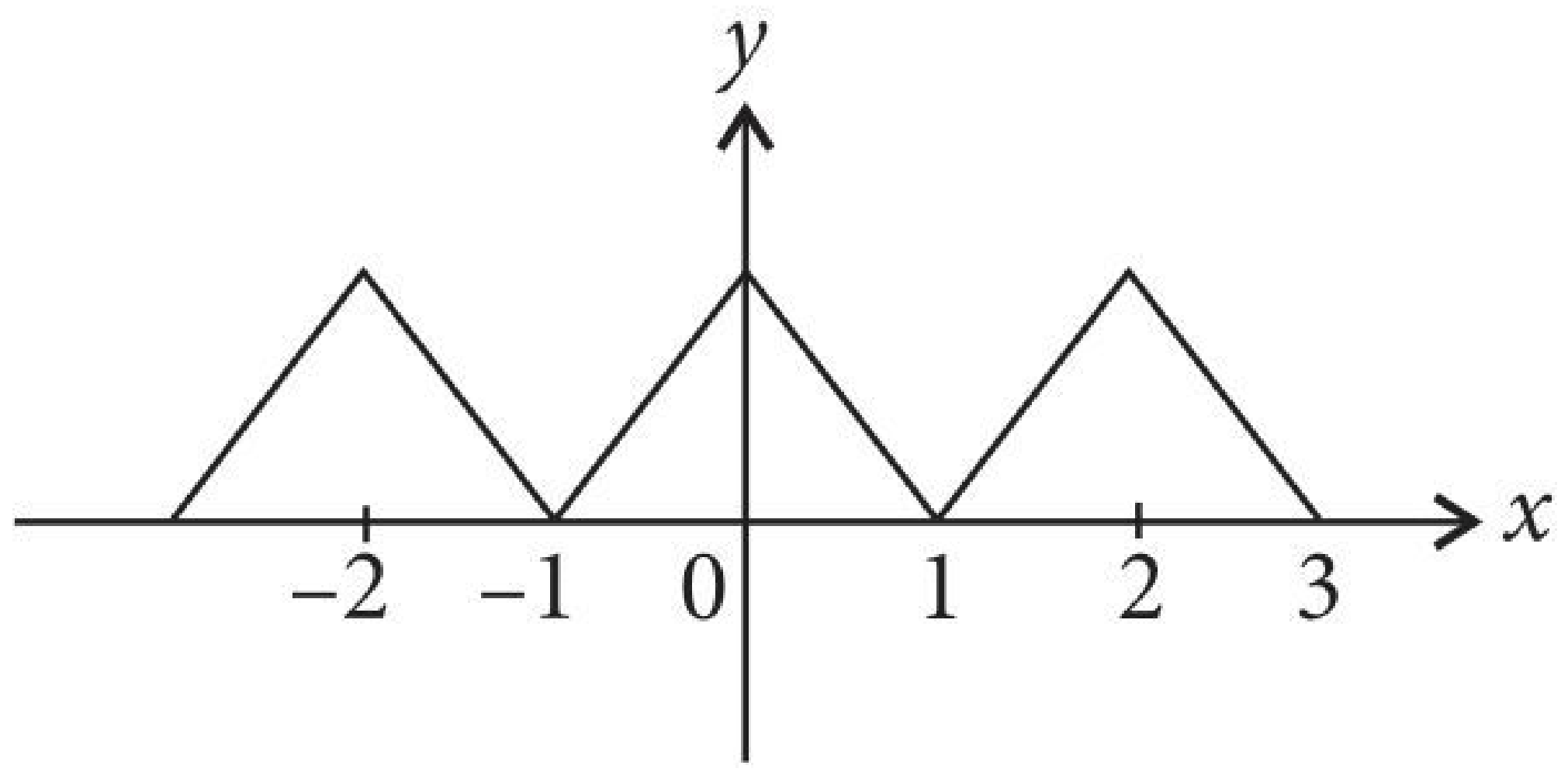
1. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x)=\begin{cases} x-[x] & \text{, if } [x] \text{ is odd} \\ 1+[x]-x & \text{, if } [x] \text{ is even}\end{cases}$$

Then find the value of $\dfrac{\pi^2}{10}\displaystyle\int_{-10}^{10} f(x)\cos \pi x\,dx$.

(Amjad, Patna)

Ans. $f(x)=\begin{cases}\{x\}, & 2n-1\le x<2n \\ 1-\{x\}, & 2n\le x<2n+1\end{cases}$



We have $f(x)$ is periodic with period 2

$$I=\int_{-10}^{10} f(x)\cos\pi x\,dx = 2\int_0^{10} f(x)\cos \pi x\,dx$$

$$=2\times 5\int_0^2 f(x)\cos\pi x\,dx$$

$$=10\left[\int_0^1 (1-x)\cos\pi x\,dx+\int_1^2 (x-1)\cos\pi x\,dx\right]$$

$$= 10\,[(I_1) + (I_2)]$$

$$I_1=\int_0^1 (1-x)\cos\pi x\,dx = -\int_0^1 x\cos\pi x\,dx$$

Also, $I_2=\displaystyle\int_1^2 (x-1)\cos\pi x\,dx$

$$=-\int_0^1 t\cos\pi t\,dt \quad (\text{Putting } x-1=t)$$

$$\therefore\quad I=10\left(-2\int_0^1 x\cos\pi x\,dx\right)$$

$$=-20\left[\frac{x\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right]_0^1$$

$$=-20\left[-\frac{1}{\pi^2}-\frac{1}{\pi^2}\right]=\frac{40}{\pi^2}$$

Hence, $\dfrac{\pi^2}{10}\displaystyle\int_{-10}^{10} f(x)\cos\pi x\,dx=\frac{\pi^2}{10}\times\frac{40}{\pi^2}=4$

2. Let $A_n=\left(\frac{3}{4}\right)-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots+(-1)^{n-1}\left(\frac{3}{4}\right)^n$

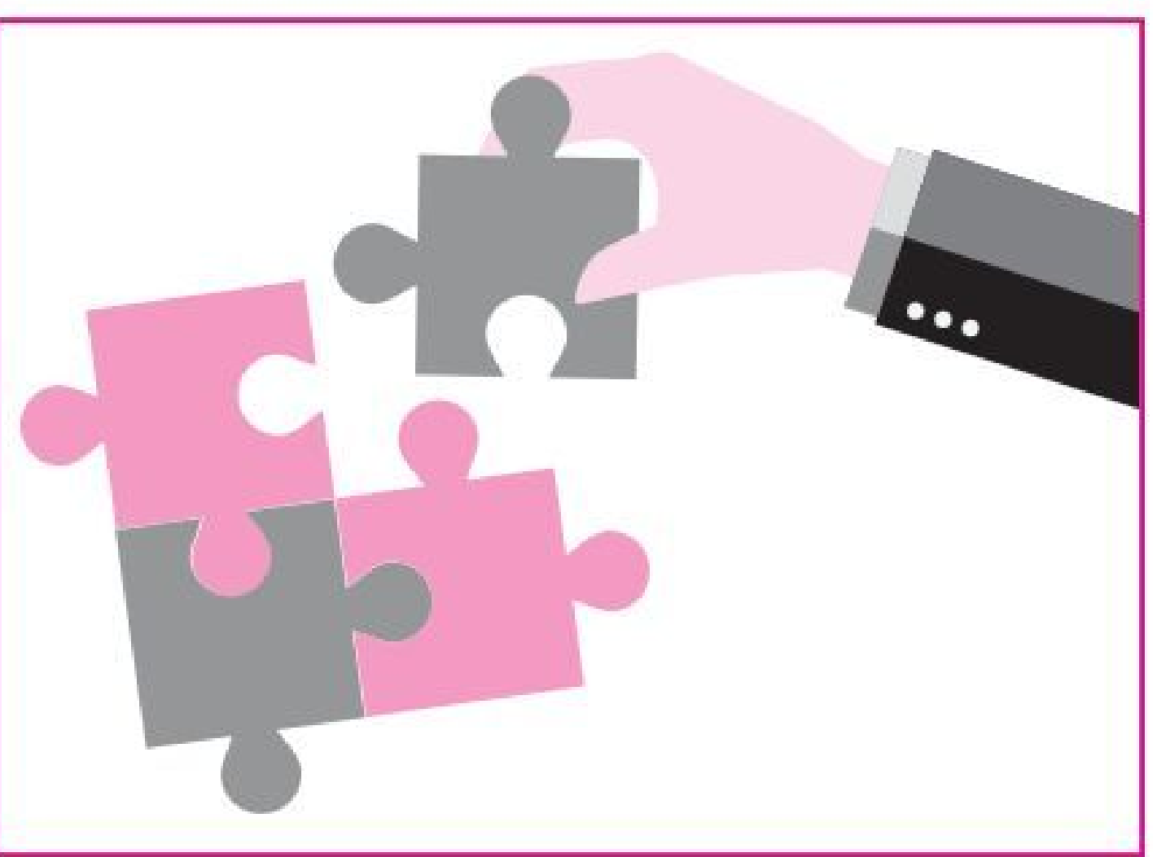
$B_n = 1 - A_n$. Find a least odd natural number n_0, so that $B_n > A_n$, $\forall\ n \ge n_0$.

(Ankush, U.P.)

Ans. $B_n = 1 - A_n > A_n \Rightarrow A_n < \frac{1}{2}$

Now, $A_n=\dfrac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}}<\dfrac{1}{2}$

PUZZLE CORNER



ANSWER · MAY 2021

12+ 1	6	8+ 4	2	5 5	10+ 3
5	3 3	2	4 4	6	1
12+ 4	5	3	9+ 1	2	6
10+ 6	2	12+ 5	3 3	1 1	9+ 4
2 2	8+ 4	1	6	7+ 3	5
3	1	11+ 6	5	4	2 2

$$\Rightarrow \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

Obviously, it is true for all even values of n

But for $n=1$, $-\frac{3}{4} < -\frac{1}{6}$

For $n=3$, $\left(\frac{-3}{4}\right)^3 = -\frac{27}{64} < -\frac{1}{6}$

For $n=5$, $\left(-\frac{3}{4}\right)^5 = -\frac{243}{1024} < -\frac{1}{6}$

For $n=7$, $\left(\frac{-3}{4}\right)^7 = \frac{-2187}{16384} > \frac{-1}{6}$

Hence, minimum odd natural number $n_0 = 7$.

3. Integrate $\sin x \cdot \sin 2x \cdot \sin 3x + \sec^2 x \cdot \cos^2 2x + \sin^4 x \cdot \cos^4 x$.

(Shekher, M.P.)

Ans. Let $I_1 = \int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x)\, dx$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24}$$

$$I_2 = \int \sec^2 x \cdot \cos^2 2x dx$$

$$= \int \sec^2 x (2\cos^2 x - 1)^2 \, dx$$

$$= \int (4\cos^2 x + \sec^2 x - 4)\, dx$$

$$= \int (2\cos 2x + \sec^2 x - 2)\, dx$$

$$= \sin 2x + \tan x - 2x$$

and $I_3 = \int \sin^4 x \cos^4 x dx$

$$= \frac{1}{128}\int (3 - 4\cos 4x + \cos 8x)\, dx$$

$$= \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$$

$$\therefore \quad I = I_1 + I_2 + I_3$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + \sin 2x + \tan x - 2x + \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}.$$

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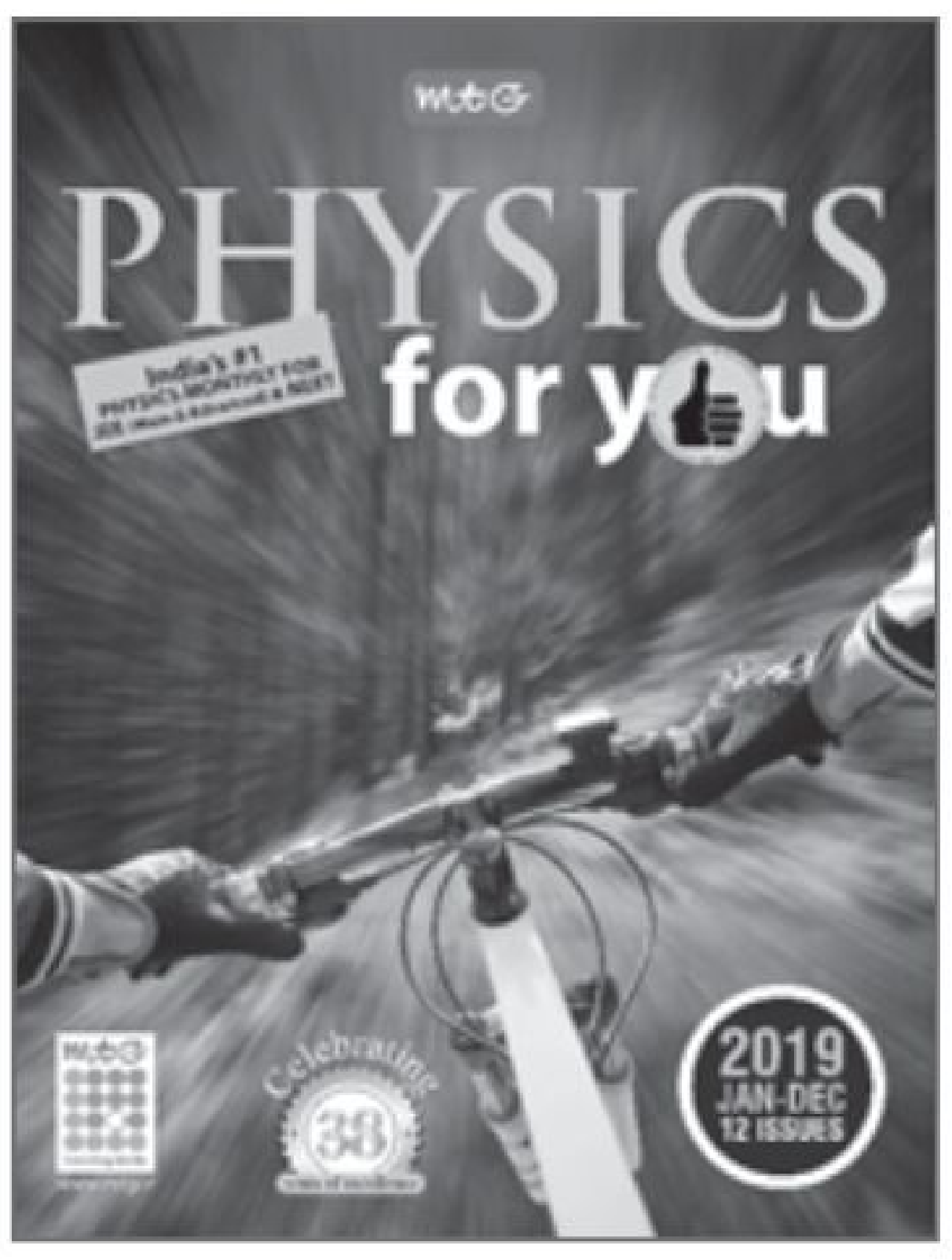

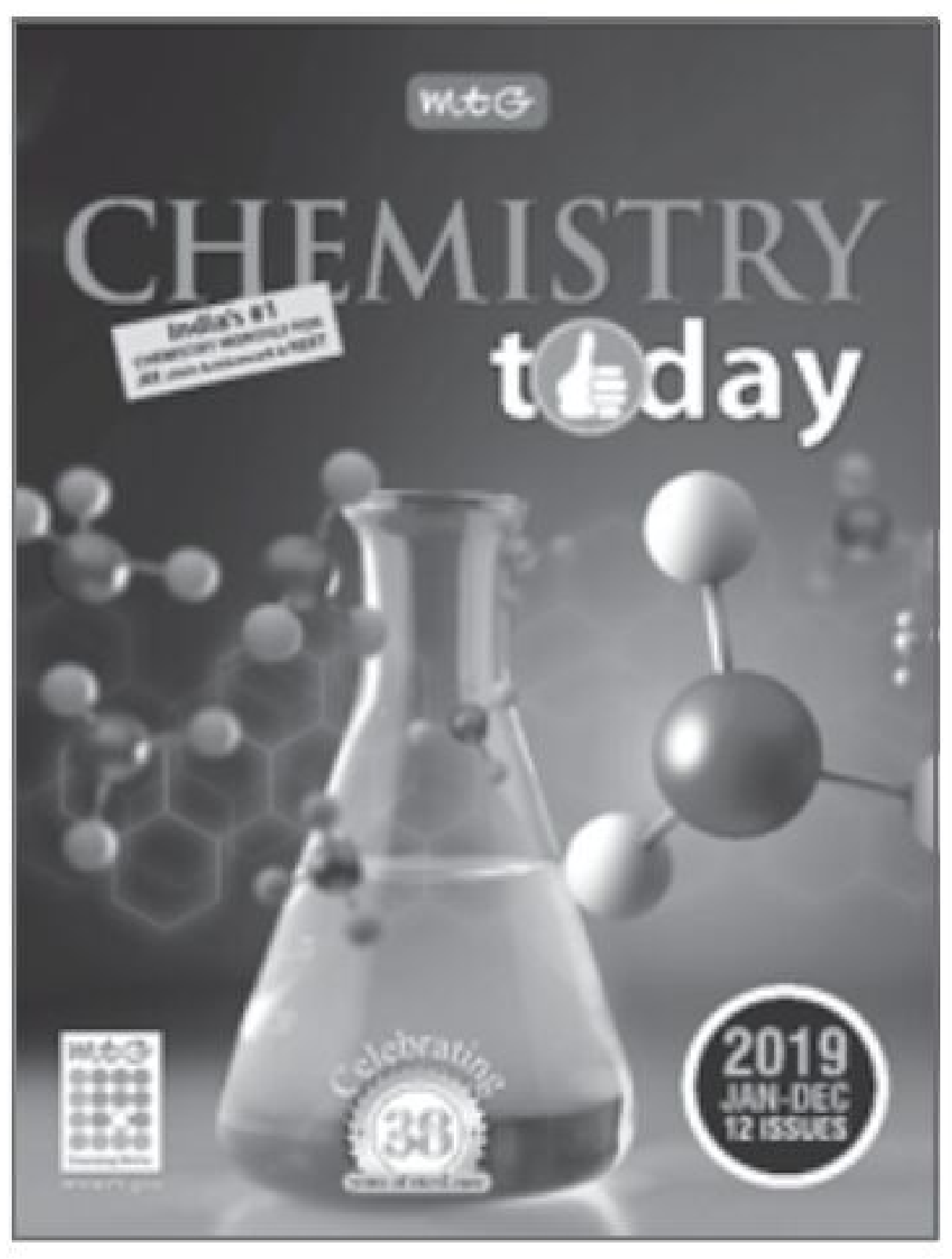

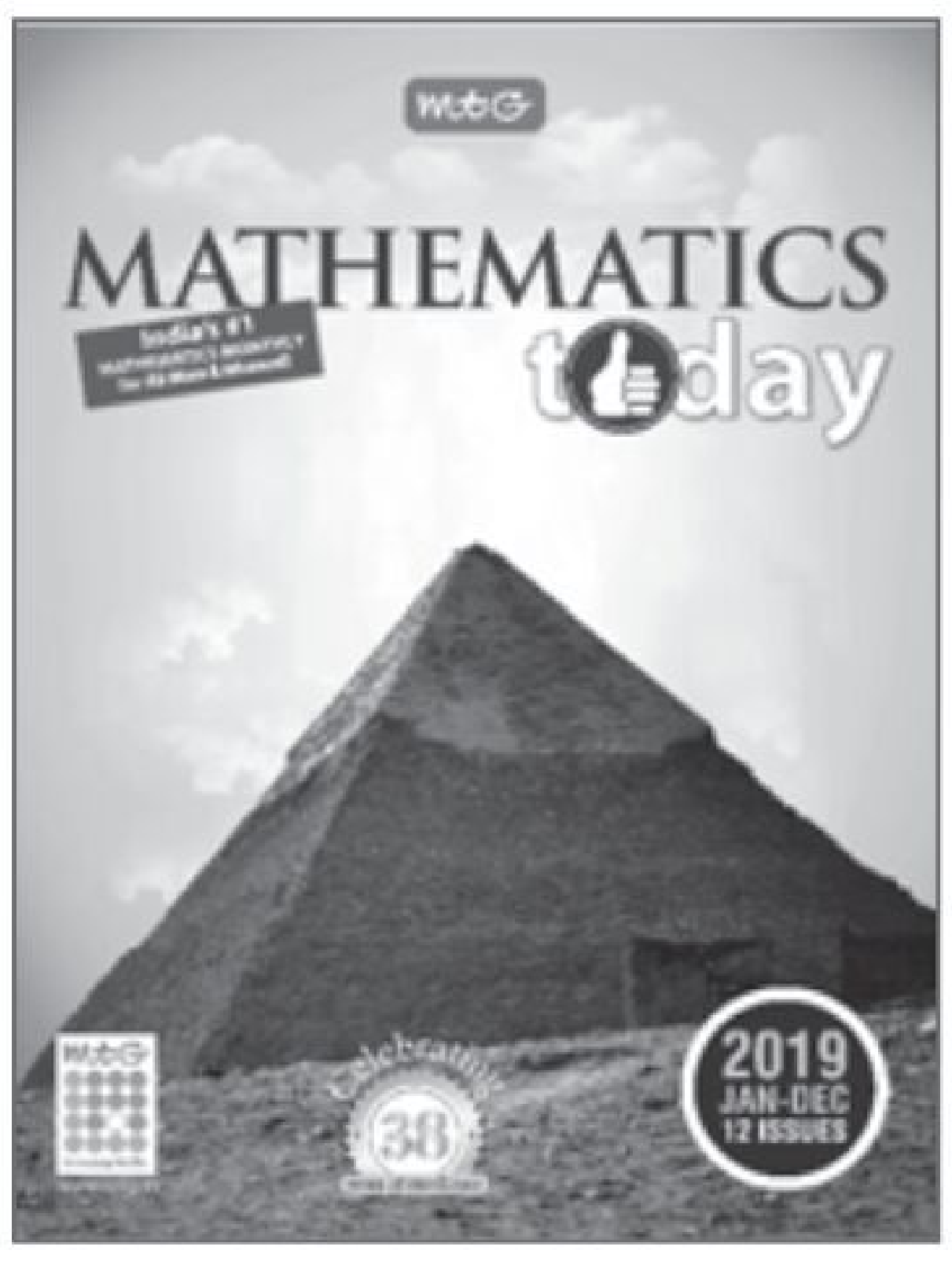

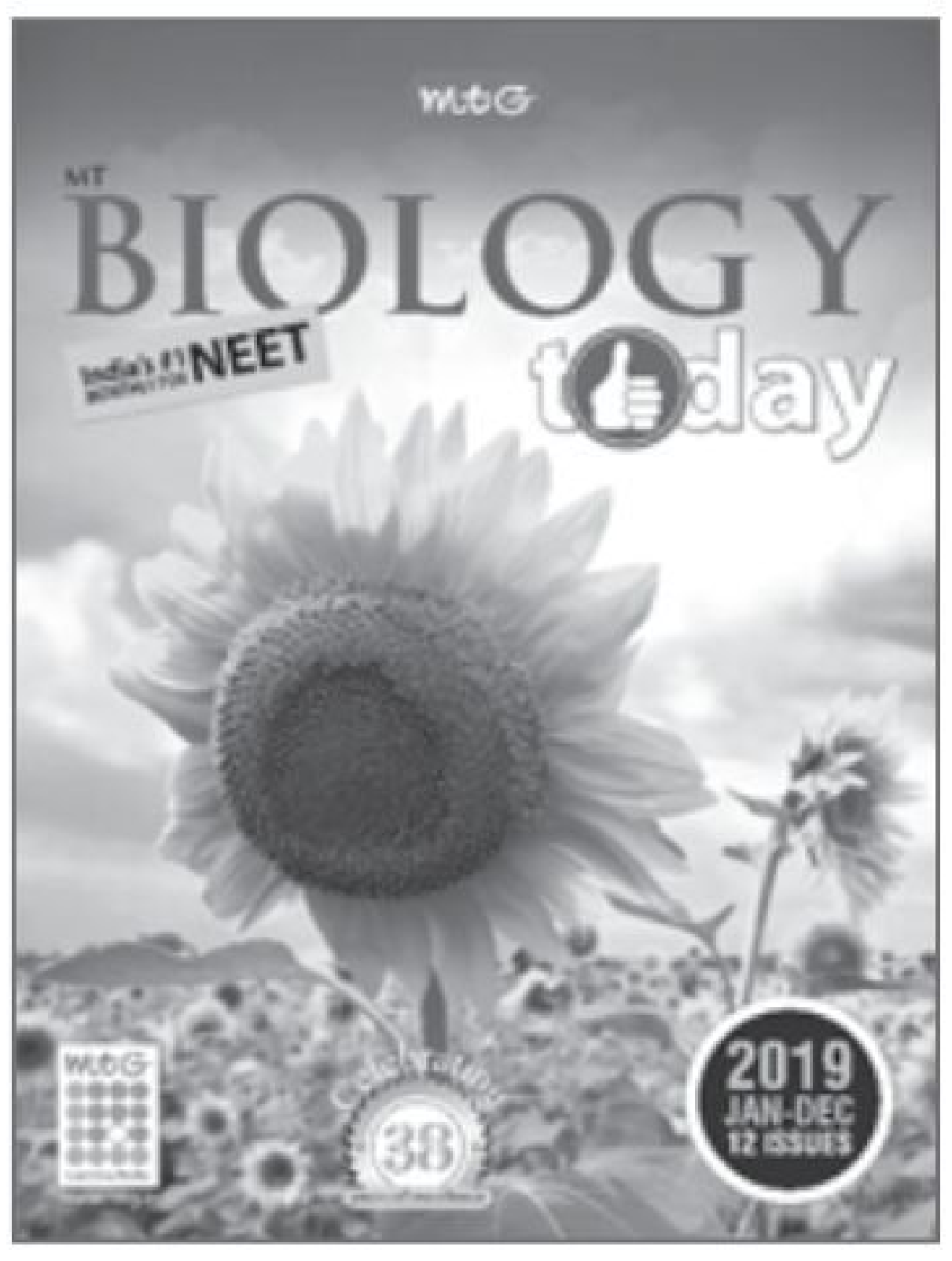

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MONTHLY TEST DRIVE

XI

This specially designed column enables students to self analyse their extent of understanding of all chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The solution of the equation $(3|x|-3)^2 = |x| + 7$ which belongs to the domain of $\sqrt{x(x-3)}$ are given by

(a) $\pm\frac{1}{9}, \pm 2$ (b) $\frac{1}{9}, -3$

(c) $-\frac{1}{9}, -2$ (d) $-\frac{1}{9}, 0$

2. In a G.P. if the sum of infinite terms is 20 and the sum of their squares is 100, then the common ratio is

(a) 5 (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

3. Let S be a non-empty subset of R. Consider the following statement:
P : There is a rational number $x \in S$ such that $x > 0$.
Which of the following statements is the negation of the statement P ?

(a) There is a rational number $x \in S$ such that $x \le 0$.
(b) There is no rational number $x \in S$ such that $x \le 0$.
(c) Every rational number $x \in S$ satisfies $x \le 0$.
(d) $x \in S$ and $x \le 0$. $\Rightarrow$ x is not rational.

4. The coefficient of x^{n-2} in the polynomial $(x-1)(x-2)(x-3) \ldots (x-n)$ is

(a) $\frac{n(n^2+2)(3n+1)}{24}$ (b) $\frac{n(n^2-1)(3n+2)}{24}$

(c) $\frac{n(n^2+1)(3n+4)}{24}$ (d) none of these

5. Different words are being formed by arranging the letters of the word "SUCCESS". All the words obtained by writing in the form of a dictionary. The number of words in which the two C are together but no two S are together, is

(a) 120 (b) 96 (c) 24 (d) 420

6. If the sides of a right angled triangle form an A.P., then the sines of the acute angles are

(a) $\frac{3}{5}, \frac{4}{5}$ (b) $\sqrt{3}, \frac{1}{3}$

(c) $\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)}, \sqrt{\left(\frac{\sqrt{5}+1}{2}\right)}$

(d) $\frac{\sqrt{3}}{2}, \frac{1}{2}$

One or More Than One Option(s) Correct Type

7. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h$, $\beta + h$ are the roots of $px^2 + qx + r = 0$, then

(a) $h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$ (b) $\frac{b^2-4ac}{a^2} = \frac{q^2-4pr}{p^2}$

(c) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ (d) none of these

8. A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then,

(a) $a = 14/15, b = 112/15$
(b) $a = 14/15, b = -8/115$
(c) $a = 64/115, b = -8/115$
(d) $a = 64/15, b = 14/15$

9. Two circles each of radius 5 units touch each other at (1, 2). If equation of their common tangent is $4x + 3y = 10$, then the equations of the circles are

(a) $x^2 + y^2 + 4x + 4y - 25 = 0$
(b) $x^2 + y^2 + 6x + 2y - 15 = 0$
(c) $x^2 + y^2 - 10x - 10y + 25 = 0$
(d) $x^2 + y^2 + 8x + 4y + 15 = 0$

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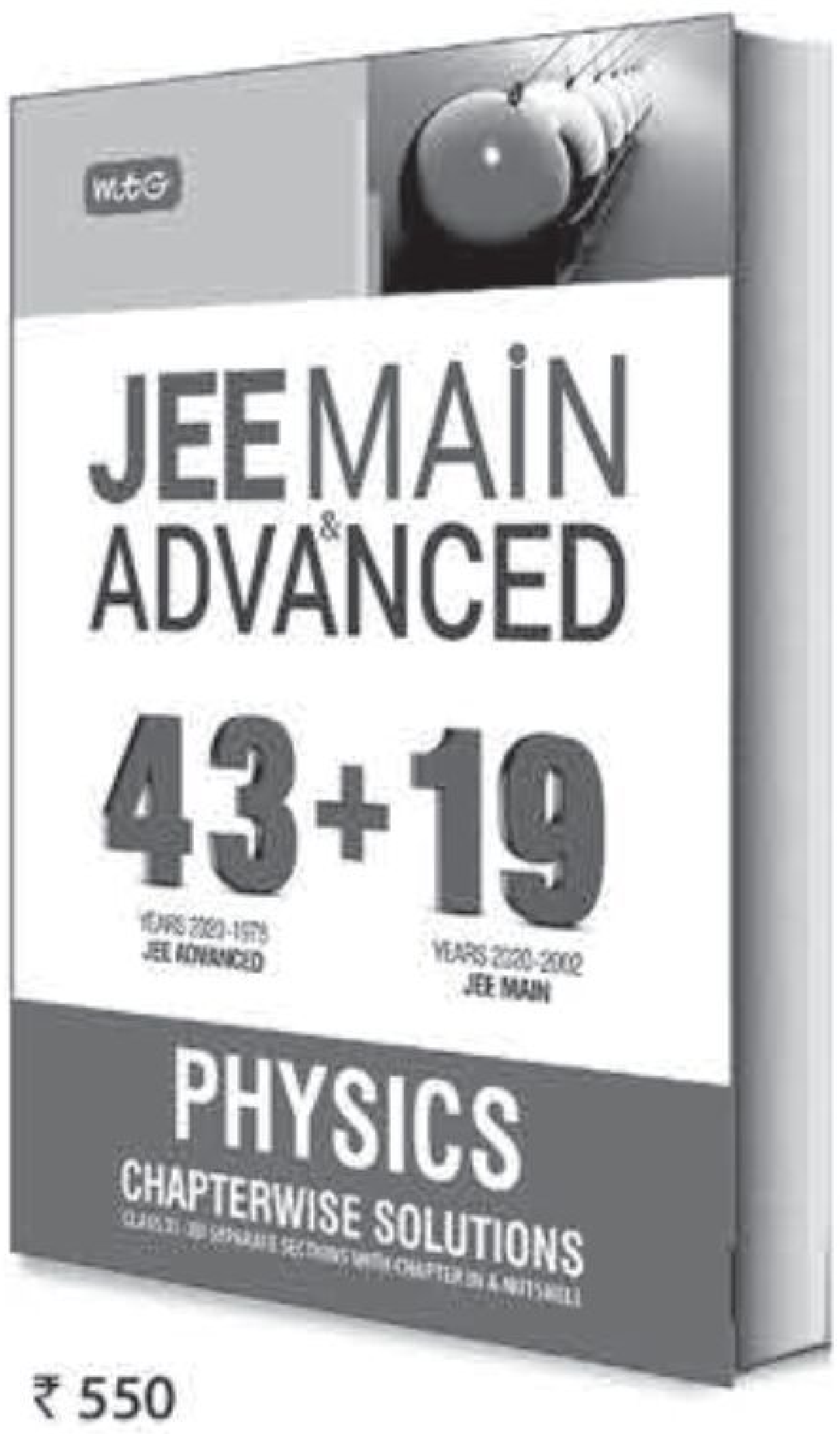
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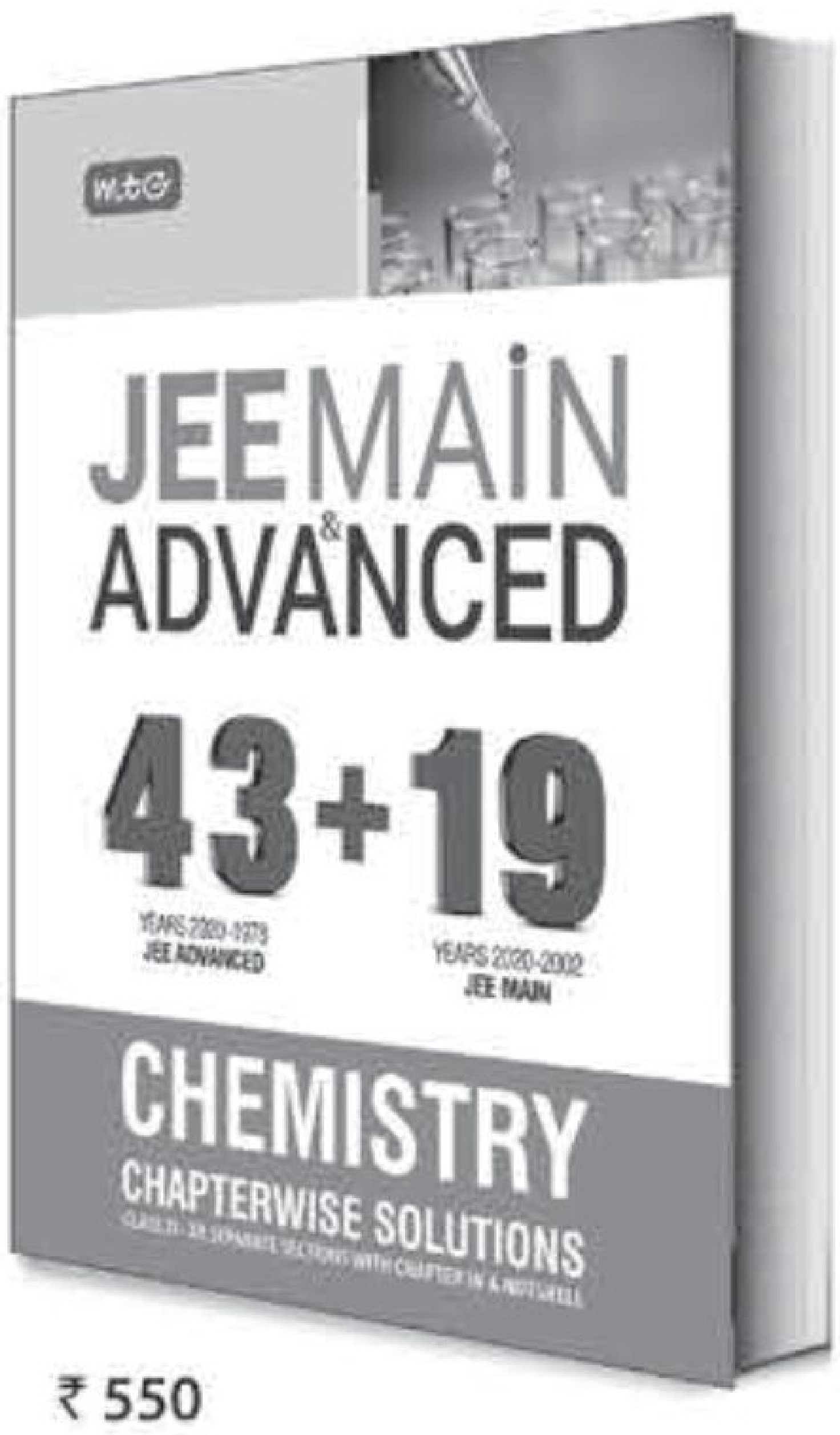
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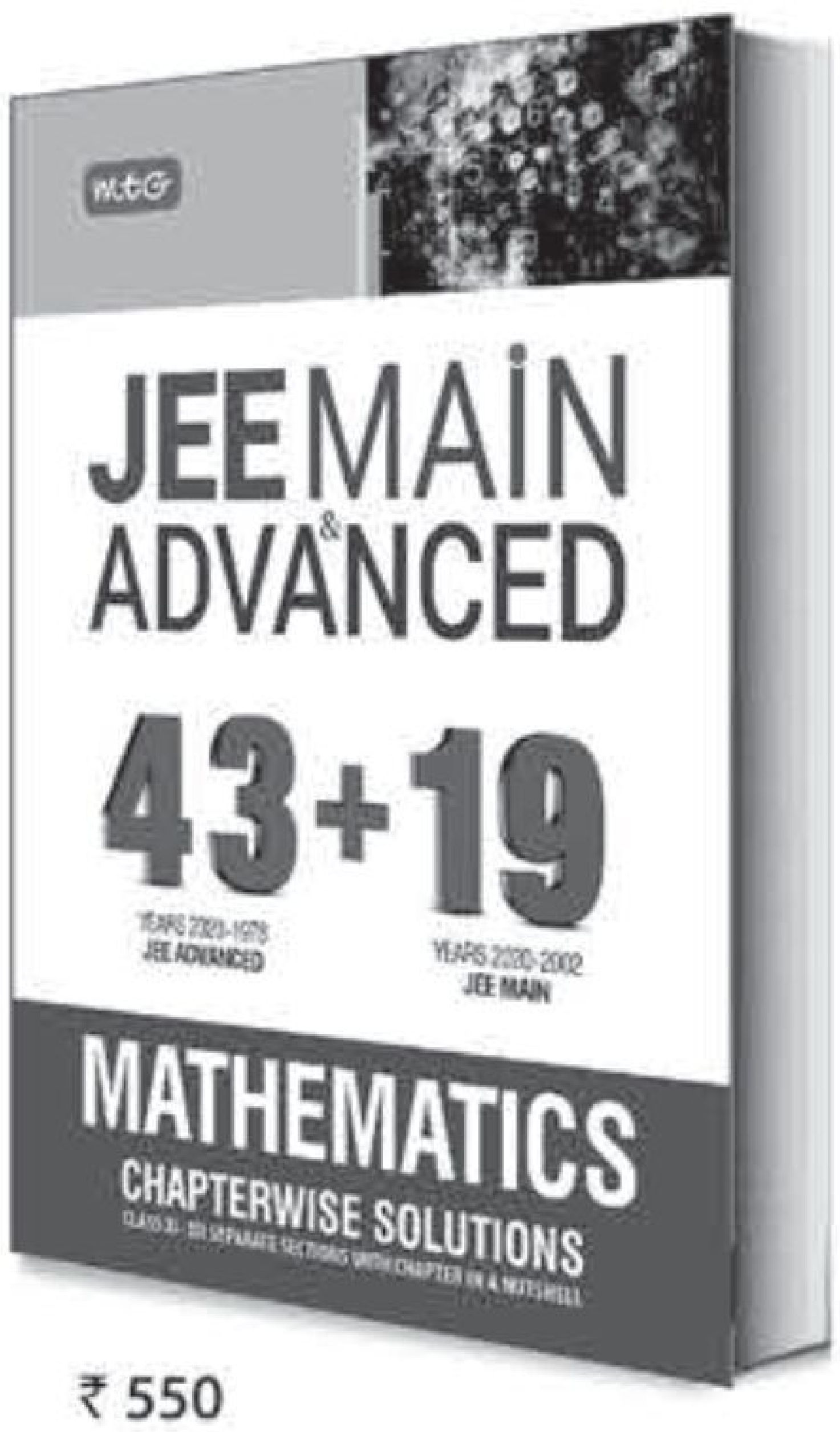
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10. If $a_i > 0$ for all $i \in N$, then

(a) $(a_1 + a_2 + a_3)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right) \geq 9$

(b) $\left(\frac{a_1}{a_2}+\frac{a_3}{a_4}+\frac{a_5}{a_6}\right)\left(\frac{a_2}{a_1}+\frac{a_4}{a_3}+\frac{a_6}{a_5}\right) \geq 9$

(c) $(a_1a_2 + a_3a_4)(a_1 a_3 + a_2a_4) \geq 4(a_1a_2a_3a_4)$

(d) none of these

11. $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11$, $0 \leq \theta \leq 4\pi$, $x \in R$, holds for

(a) no value of x and θ

(b) one value of x and two values of θ

(c) two values of x and two values of θ

(d) none of these

12. The number of ways in which we can arrange the $2n$ letters $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ in a line so that the suffixes of letters x and those of the letters y are respectively in ascending order of magnitude is

(a) $({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2$

(b) ${}^{2n}C_n$

(c) $2^n[1 \cdot 3 \cdot 5 \ldots (2n - 1)]/n!$

(d) ${}^{2n}C_n - 1$

13. The ends of a line segment are P (1, 3) and Q (1, 1). R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$, then

(a) $\lambda \in (0, 1)$

(b) $\lambda \in \left(-\frac{3}{5}, 1\right)$

(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$

(d) none of these

Comprehension Type

ABC is a triangle right angled at A, $AB = 2AC$. $A = (1, 2)$, $B = (-3, 1)$. ACD is an equilateral triangle. The vertices of the two triangles are in anti-clockwise sense. $BCEF$ is a square with vertices in clockwise sense.

14. Area of ΔACF =

(a) $\frac{51}{8}$ (b) $\frac{51}{4}$ (c) $\frac{31}{5}$ (d) $\frac{21}{4}$

15. DE =

(a) $\sqrt{17}\sqrt{4-\sqrt{3}}$

(b) $\frac{\sqrt{17}}{2}\sqrt{8+\sqrt{3}}$

(c) $\frac{\sqrt{17}}{2}\sqrt{4+\sqrt{3}}$

(d) $\sqrt{15}\sqrt{4+\sqrt{3}}$

Matrix Match Type

16. Match the following :

Column-I		Column-II	
P.	If $\theta+\phi=\frac{\pi}{2}$, where θ and ϕ are positive, then maximum value of $(\sin\theta + \sin\phi)\sin\left(\frac{\pi}{4}\right)$ is	1.	1
Q.	If $\sin\theta - \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then greatest value of $a^2 + b^2$ is	2.	13
R.	If $3\sin\theta + 5\cos\theta = 5$, $(\theta \neq 0)$, then the value of $5\sin\theta - 3\cos\theta$ is	3.	3
S.	If $\cos\alpha + \cos\beta = \frac{1}{2}$, $\sin\alpha + \sin\beta = \frac{1}{3}$ and $\tan^2\left(\frac{\alpha-\beta}{2}\right) = \frac{131}{K}$, then K is	4.	4

	P	Q	R	S
(a)	2	4	3	1
(b)	1	3	4	2
(c)	1	4	3	2
(d)	3	4	2	1

Numerical Value Type

17. If the area of the triangle on the complex plane formed by the points z, $z + iz$ and iz is 200, then the value of $\frac{|z|}{5}$ must be equal to (where $i = \sqrt{-1}$) ________.

18. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Then, the value of b^2 is________.

19. If the integer k is added to each of the numbers, 36, 300, 596, one obtains the squares of three consecutive terms of an A.P., then the last digit of k is________.

20. If $\log_{245} 175 = a$, $\log_{1715} 875 = b$, then the value of $\frac{1-ab}{a-b}$ is________.

◆ ◆

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	**EXCELLENT WORK !**	You are well prepared to take the challenge of final exam.
90-75%	**GOOD WORK !**	You can score good in the final exam.
74-60%	**SATISFACTORY !**	You need to score more next time.
< 60%	**NOT SATISFACTORY!**	Revise thoroughly and strengthen your concepts.

CBSE warm-up!

CLASS-XII

Practice paper for CBSE Exam as per the reduced syllabus
and marking scheme issued by CBSE for the academic session 2020-21.

Practice Paper 2021

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(1) This question paper contains two parts A and B. Each part is compulsory. Part A carries 24 marks and Part B carries 56 marks.
(2) Part-A has Objective Type Questions and Part -B has Descriptive Type Questions.
(3) Both Part A and Part B have internal choices.

Part-A :

(1) It consists of two sections- I and II.
(2) Section I comprises of 16 very short answer type questions.
(3) Section II contains 2 case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt any 4 out of 5 MCQs.

Part-B :

(1) It consists of three sections- III, IV and V.
(2) Section III comprises of 10 questions of 2 marks each.
(3) Section IV comprises of 7 questions of 3 marks each.
(4) Section V comprises of 3 questions of 5 marks each.
(5) Internal choice is provided in 3 questions of Section-III, 2 questions of Section-IV and 3 questions of Section-V. You have to attempt only one of the alternatives in all such questions.

PART-A

SECTION - I

All questions are compulsory. In case of internal choices attempt any one.

1. Show that the function $f : N \to N$ given by $f(x) = 4x$, is one-one but not onto.

OR

Let $f : R \to R$ be a function defined by $f(x) = x^3 + 4$, then check whether f is a bijection or not.

2. What positive value of x makes the following pair of determinants equal ?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

3. If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that $2A + X = 5B$.

OR

Simplify :

$$\tan\theta\begin{bmatrix}\sec\theta & \tan\theta\\ \tan\theta & -\sec\theta\end{bmatrix}+\sec\theta\begin{bmatrix}-\tan\theta & -\sec\theta\\ -\sec\theta & \tan\theta\end{bmatrix}$$

4. What is the degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^{2/3}+4-\frac{3dy}{dx}=0\ ?$$

5. If a line makes angles 90°, 60° and 30° with the positive directions of x, y and z-axis respectively, then find its direction cosines.

OR

Find the direction cosines of the line passing through two points (2, 1, 0) and (1, –2, 3).

6. If $f(x)=\begin{vmatrix}(1+x)^{17} & (1+x)^{19} & (1+x)^{23}\\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34}\\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47}\end{vmatrix}=A$ $+ Bx + Cx^2 + \ldots$, then prove that $A = 0$.

7. Evaluate : $\int_0^{\pi/2} x\cos x\, dx$

OR

Evaluate : $\int \cos^3 x \sin x\, dx$

8. Find the angle between the vectors $\vec a$ and $\vec b$ if $\vec a = 2\hat i - \hat j + 2\hat k$ and $\vec b = 4\hat i + 4\hat j - 2\hat k$.

9. A card is picked at random from a pack of 52 playing cards. Given that the picked card is a queen. Find the probability of this card to be a card of spade.

OR

A die is thrown once. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then find $P(A \cup B)$.

10. Let $R = \{(a, a^3) : a$ is a prime number less than $5\}$ be a relation. Find the range of R.

11. Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2$ divides $(a - b)\}$. Write the equivalence class [0].

12. Find the area bounded by $y = x^2$, the x- axis and the lines $x = -1$ and $x = 1$.

13. Write the position vector of the point which divides the join of points with position vectors $3\vec a - 2\vec b$ and $2\vec a + 3\vec b$ in the ratio 2 : 1.

14. Find the area of a parallelogram whose adjacent sides are represented by the vectors $2\hat i - 3\hat k$ and $4\hat j + 2\hat k$.

15. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.

16. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then find $P(B'|A)$.

SECTION - II

Both the case study based questions are compulsory. Attempt any 4 sub-parts from each question 17 and 18. Each sub-part carries 1 mark.

17. In a family there are four children. All of them have to work in fields to earn their livelihood at the age of 15.

Based on the above information, answer the following questions :

(i) Probability that all children working in fields are boys if it is given that elder child working in fields is a boy, is

(a) 3/8 (b) 1/8
(c) 5/8 (d) none of these

(ii) Probability that all children working in fields are girls if first two children working in fields are girls, is

(a) 1/4 (b) 3/4
(c) 1/2 (d) none of these

(iii) Find the probability that two middle child working in fields are boys if it is given that first child working in fields is a girl.

(a) 0 (b) 3/4
(c) 1/4 (d) none of these

(iv) Find the probability that all children working in fields are girls if it is given that at least one of the children working in fields is a girl.

(a) 0 (b) 1/15
(c) 2/15 (d) 4/15

(v) Find the probability that all children working in fields are boys if it is given that at least three of the children working in fields are boys.

(a) 1/5 (b) 2/5

(c) 3/5 (d) 4/5

18. In a street two lamp posts are 300 feet apart. The light intensity at a distance d from the first (stronger) lamp post is $\frac{1000}{d^2}$, the light intensity at distance d from the second (weaker) lamp post is $\frac{125}{d^2}$ (in both cases the light intensity is inversely proportional to the square of the distance to the light source). The combined light intensity is the sum of the two light intensities coming from both lamp posts.

Based on the above information, answer the following.

(i) If you are in between the lamp posts, at distance x feet from the stronger light, then the formula for the combined light intensity coming from both lamp posts as function of x, is

(a) $\frac{1000}{x^2}+\frac{125}{x^2}$ (b) $\frac{1000}{(300-x^2)}+\frac{125}{x^2}$

(c) $\frac{1000}{x^2}+\frac{125}{(300-x)^2}$

(d) None of these

(ii) The maximum value of x can not be

(a) 100 (b) 200

(c) 300 (d) None of these

(iii) The minimum value of x can not be

(a) 0 (b) 100

(c) 200 (d) None of these

(iv) If $I(x)$ denote the combined light intensity, then $I(x)$ will be minimum when x =

(a) 100 (b) 200

(c) 300 (d) 150

(v) The darkest spot between the two lights is

(a) at a distance of 100 feet from the weaker lamp post.

(b) at distance of 100 feet from the stronger lamp post.

(c) at a distance of 200 feet from the weaker lamp post.

(d) None of these

PART-B

SECTION-III

All questions are compulsory. In case of internal choices attempt any one.

19. If $0 < x < 1$, then

$$\sqrt{1+x^2}\left[(x\cos[\cot^{-1}x]+\sin[\cot^{-1}x])^2-1\right]^{1/2}.$$

20. If $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}-\hat{k}$, then find $(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b})$.

OR

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}, \vec{c}=\hat{i}$ and $(\vec{a}\times\vec{b})\times\vec{c}$ $=\lambda\vec{a}+\mu\vec{b}$, then find the value of $\lambda + \mu$.

21. Two cards are drawn successively, without replacement, from a well-shuffled pack of 52 cards. Find the probability distribution of number of spades.

22. Show that the function f given by $f(x) = x^3 - 3x^2 + 4x$, $x \in R$ is increasing on R.

23. Given $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$, compute A^{-1}, if exists.

OR

Find a matrix A such that $2A - 3B + 5C = O$, where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$.

24. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

25. Find the coordinates of the point where the line through $(-1, 1, -8)$ and $(5, -2, 10)$ crosses the ZX-plane.

26. Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$.

27. Solve the following differential equation :

$$\frac{dy}{dx}+y=\cos x-\sin x$$

28. Find $\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$.

OR

Evaluate : $\int_0^4(|x|+|x-2|+|x-4|)dx$

SECTION-IV

All questions are compulsory. In case of internal choices attempt any one.

29. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for (a, b), (c, d) in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$.

30. Find the relationship between a and b so that the function 'f' defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

31. Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not.

$$f(x) = \begin{cases} x, & x < 1 \\ 2-x, & 1 \le x \le 2 \\ -2+3x-x^2, & x > 2 \end{cases}$$

OR

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$

32. Find the equations of tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

33. Find the area enclosed by $y = 3x - 5$, $y = 0$, $x = 3$ and $x = 5$.

34. Evaluate : $\int \frac{1 + x\cos x}{x(1 - x^2 e^{2\sin x})} dx$

OR

Evaluate : $\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

35. Find the solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$.

SECTION-V

All questions are compulsory. In case of internal choices attempt any one.

36. Solve the following linear programming problem (LPP) graphically.

Maximize $Z = 4x + 6y$

Subject to constraints:

$x + 2y \le 80,\ 3x + y \le 75\ ;\ x, y \ge 0$

OR

Solve the following linear programming problem (LPP) graphically.

Minimize $Z = 30x + 20y$

Subject to constraints : $x + y \le 8$, $x + 4y \ge 12$, $5x + 8y \ge 20\ ;\ x, y \ge 0$

37. Using matrices, solve the following system of equations :

$x + y - z = 3$; $2x + 3y + z = 10$;
$3x - y - 7z = 1$

OR

If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ are two square matrices, find AB and hence solve the system of equations

$x - y = 3$; $2x + 3y + 4z = 17$; $y + 2z = 7$.

38. Find the length and the foot of the perpendicular from the point $P(7, 14, 5)$ to the plane $2x + 4y - z = 2$. Also find the image of point P in the plane.

OR

Find the vector equation of a line passing through the point $(2, 3, 2)$ and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between these two lines.

SOLUTIONS

1. For one-one : Consider, $f(x_1) = f(x_2)$

$\Rightarrow\ 4x_1 = 4x_2 \Rightarrow x_1 = x_2$

Hence, f is one-one.

For onto : Let y be any element in N(co-domain), then $f(x) = y \Rightarrow\ 4x = y$

$\Rightarrow\ x = \frac{y}{4}$. But $\forall y \in N$, $\frac{y}{4} \notin N$

Thus, $f(x)$ is not onto.

OR

Given $f(x) = x^3 + 4$. Let $x_1, x_2 \in R$

For one - one : Consider, $f(x_1) = f(x_2) \Rightarrow x_1^3 + 4 = x_2^3 + 4$

$\Rightarrow\ x_1^3 = x_2^3 \Rightarrow\ x_1 = x_2$

$\therefore\ f(x)$ is one-one.

For onto : Let y be any element in R (co-domain), then $f(x) = y \Rightarrow x^3 + 4 = y$

$\Rightarrow\ x = \sqrt[3]{y-4} \in R$ (domain)

$\therefore\ f(x)$is onto.

Hence, it is a bijection.

2. We have, $\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

$\Rightarrow$ $2x^2 - 15 = 32 - 15 \Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16$

$\Rightarrow$ $x = 4$ $[\because x > 0]$

3. We have, $2A + X = 5B$

$\Rightarrow$ $X = 5B - 2A$

$$\Rightarrow X = \left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 10-16 & -10-0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix}$$

OR

We have,

$$\tan\theta\begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & -\sec\theta \end{bmatrix} + \sec\theta\begin{bmatrix} -\tan\theta & -\sec\theta \\ -\sec\theta & \tan\theta \end{bmatrix}$$

$$= \begin{bmatrix} \tan\theta\sec\theta & \tan^2\theta \\ \tan^2\theta & -\tan\theta\sec\theta \end{bmatrix} + \begin{bmatrix} -\tan\theta\sec\theta & -\sec^2\theta \\ -\sec^2\theta & \tan\theta\sec\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

4. We have, $\left(\dfrac{d^2y}{dx^2}\right)^{2/3} = 3\dfrac{dy}{dx} - 4$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(3\frac{dy}{dx} - 4\right)^3 \quad \text{(On cubing both sides)}$$

$\therefore$ Degree is 2.

5. Let the direction cosines of the line be l, m, n. Then, $l = \cos 90° = 0$, $m = \cos 60° = \dfrac{1}{2}$ and $n = \cos 30° = \dfrac{\sqrt{3}}{2}$

So, direction cosines are $< 0, \dfrac{1}{2}, \dfrac{\sqrt{3}}{2} >$.

OR

Here, $P(2, 1, 0)$ and $Q(1, -2, 3)$

So, $PQ = \sqrt{(1-2)^2 + (-2-1)^2 + (3-0)^2}$

$= \sqrt{1 + 9 + 9} = \sqrt{19}$

Thus, the direction cosines of the line joining two points are $< \dfrac{1-2}{\sqrt{19}}, \dfrac{-2-1}{\sqrt{19}}, \dfrac{3-0}{\sqrt{19}} >$ *i.e.*, $< \dfrac{-1}{\sqrt{19}}, \dfrac{-3}{\sqrt{19}}, \dfrac{3}{\sqrt{19}} >$

6. Given that,

$$f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$$

$= A + Bx + Cx^2 + \ldots..$

On putting $x = 0$, we have $f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = A$

$\Rightarrow$ $A = 0$ $\quad$ ($\because$ R_1 and R_2 are identical)

7. $\displaystyle\int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx$

[Integrating by parts]

$$= \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$$

OR

We have, $\int \cos^3 x \sin x\,dx$

Put $\cos x = t \Rightarrow \sin x\,dx = -dt$

$$\therefore \int \cos^3 x \sin x\,dx = -\int t^3 dt = -\frac{t^4}{4} + C$$

$$= -\frac{1}{4}\cos^4 x + C$$

8. We have, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$

Now, $\vec{a}\cdot\vec{b} = (2\hat{i} - \hat{j} + 2\hat{k})\cdot(4\hat{i} + 4\hat{j} - 2\hat{k})$

$= 8 - 4 - 4 = 0$

So, angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{2}$.

9. Let A be the event that the card is a spade and B be the event that the picked card is a queen.

We have a total of 13 spades and 4 queen cards.

Also only one queen is from spade.

$$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{52}}{\frac{4}{52}} = \frac{1}{4}$$

OR

Here, $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4\}$

$A \cap B = \{4\}$

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Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = 1$$

10. Given relation is

$R = \{(a, a^3) : a \text{ is a prime number less than } 5\}$.

$\therefore \quad R = \{(2, 8), (3, 27)\}$

So, the range of R is $\{8, 27\}$.

11. Here, $R = \{(a, b) \in A \times A : 2 \text{ divides } (a - b)\}$

This is the given equivalence relation, where

$A = \{0, 1, 2, 3, 4, 5\}$

$\therefore$ Equivalence class of $[0]$ is $\{0, 2, 4\}$.

12. Required area, $A = \int_{-1}^{1} x^2\, dx$

$$\Rightarrow \quad A = 2\int_0^1 x^2 dx = \frac{2}{3}\left[x^3\right]_0^1 = \frac{2}{3} \text{ sq. units}$$

13. Required position vector

$$= \frac{2(2\vec{a} + 3\vec{b}) + 1(3\vec{a} - 2\vec{b})}{2+1} = \frac{7\vec{a} + 4\vec{b}}{3} = \frac{7}{3}\vec{a} + \frac{4}{3}\vec{b}$$

14. Let $\vec{a} = 2\hat{i} - 3\hat{k}$ and $\vec{b} = 4\hat{j} + 2\hat{k}$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 12\hat{i} - 4\hat{j} + 8\hat{k}$$

$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{(12)^2 + (-4)^2 + (8)^2} = \sqrt{144 + 16 + 64}$$

$= \sqrt{224} = 4\sqrt{14}$ sq. units.

15. Length of the ⊥ from $O(0, 0, 0)$ to the plane $2x - 3y + 6z + 21 = 0$ is

$$\left|\frac{2 \times 0 - 3 \times 0 + 6 \times 0 + 21}{\sqrt{2^2 + (-3)^2 + 6^2}}\right| = \frac{21}{7} = 3 \text{ units}.$$

16. Given, A and B are independent events.

Also, $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$

Now, $P(B'|A) = \dfrac{P(B' \cap A)}{P(A)}$

$= \dfrac{P(B')P(A)}{P(A)}$ $\quad [\because A, B$ are independent events$]$

$= P(B') = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$

17. Let B and G denote the boy and girl respectively. If a family has 4 children each of four children can either boy or girl.

Sample space is given by

$S = \{BBBB, BBBG, BBGB, BGBB, BBGG, BGBG, BGGB, BGGG, GBBB, GBBG, GBGB, GBGG, GGBB, GGBG, GGGB, GGGG\}$

(i) (b) : Let A = All children are boys.

$\therefore A = \{BBBB\}$ *i.e.*, $n(A) = 1$

B = Elder child is a boy

$\therefore B = \{BBBB, BBBG, BBGB, BGBB, BBGG, BGBG, BGGB, BGGG\}$ *i.e.*, $n(B) = 8$

Now, $n(A \cap B) = 1$

$$\therefore P(A/B) = \frac{(A \cap B)}{n(B)} = \frac{1}{8}$$

(ii) (a) : Let A = All are girls.

$\therefore \quad A = \{GGGG\}$ *i.e.*, $n(A) = 1$

B = First two children are girl

$\therefore \quad B = \{GGBB, GGBG, GGGB, GGGG\}$ *i.e.*, $n(B) = 4$

Now, $n(A \cap B) = 1$

$$\therefore P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{4}$$

(iii) (c) : Let A = Two middle child are boys.

$\therefore \quad A = \{BBBB, BBBG, GBBB, GBBG\}$ *i.e.*, $n(A) = 4$

B = First child is a girl = 8

$\therefore \quad B = \{GBBB, GBBG, GBGB, GBGG, GGBB, GGBG, GGGB, GGGG\}$

i.e., $n(B) = 8$

Now, $n(A \cap B) = 2$

$$\therefore P(A/B) = \frac{2}{8} = \frac{1}{4}$$

(iv) (b) : Let A = All are girls.

$\therefore \quad A = \{GGGG\}$ *i.e.* $n(A) = 1$

B = At least one child is girl.

$\therefore \quad B = \{BBBG, BBGB, BGBB, GBBB, BBGG, GGBB, GBGB, BGBG, BGGB, GBBG, GGGB, GGBG, GBGG, BGGG, GGGG\}$

i.e., $n(B) = 15$

Now, $n(A \cap B) = 1$

$$\therefore P(A/B) = \frac{1}{15}$$

(v) (a) : Let A = All are boys.

$\therefore \quad A = \{B\,B\,B\,B\}$ *i.e.* $n(A) = 1$

B = At least three of the children are boys.

$\therefore \quad B = \{BBBB, BBBG, BBGB, BGBB, GBBB\}$ *i.e.*, $n(B) = 5$

Now, $n(A\cap B) = 1$

$\therefore\ P(A/B)=\dfrac{1}{5}$

18. (i) (c) : Since, the distance is x feet from the stronger light, therefore the distance from the weaker light will be $300 - x$.
So, the combined light intensity from both lamp posts is given by $\dfrac{1000}{x^2}+\dfrac{125}{(300-x)^2}$.

(ii) (c) : Since, the person is in between the lamp posts, therefore x will lie in the interval (0, 300).
So, maximum value of x can't be 300.

(iii) (a) : Since, $0 < x < 300$, therefore minimum value of x can't be 0.

(iv) (b) : We have, $I(x)=\dfrac{1000}{x^2}+\dfrac{125}{(300-x)^2}$

$\Rightarrow\ I'(x)=\dfrac{-2000}{x^3}+\dfrac{250}{(300-x)^3}$ and

$\Rightarrow\ I''(x)=\dfrac{2000}{x^4}+\dfrac{750}{(300-x)^4}$

For maxima/minima, $I'(x) = 0$

$\Rightarrow\ \dfrac{2000}{x^3}=\dfrac{250}{(300-x)^3}\ \Rightarrow\ 8(300-x)^3=x^3$

Taking cube root on both sides, we get

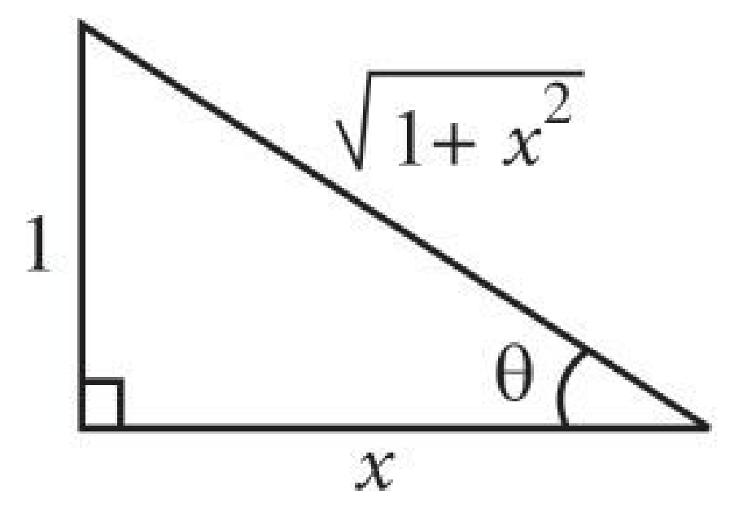
$2(300 - x) = x \Rightarrow 600 = 3x \Rightarrow x = 200$

Thus, $I(x)$ is minimum when you are at 200 feet from the strong intensity lamp post.

(v) (a) : Since, $I(x)$ is minimum when $x = 200$ feet, therefore the darkest spot between the two light is at a distance of 200 feet from stronger lamp post, *i.e.*, at a distance of 300 – 200 = 100 feet from the weaker lamp post.

19. Let $\theta = \cot^{-1}x \Rightarrow \cos(\cot^{-1}x)=\dfrac{x}{\sqrt{1+x^2}}$

and $\sin(\cot^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$



The given expression becomes

$$\sqrt{1+x^2}\left[\left(\frac{x^2+1}{\sqrt{1+x^2}}\right)^2-1\right]^{1/2}=x\sqrt{1+x^2}.$$

20. Here, $\vec{a}+3\vec{b}=\hat{i}+\hat{j}+2\hat{k}+3(3\hat{i}+2\hat{j}-\hat{k})$
$=10\hat{i}+7\hat{j}-\hat{k}$

and $2\vec{a}-\vec{b}=2(\hat{i}+\hat{j}+2\hat{k})-(3\hat{i}+2\hat{j}-\hat{k})=-\hat{i}+5\hat{k}$

$\therefore\ (\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b})=(10\hat{i}+7\hat{j}-\hat{k})\cdot(-\hat{i}+5\hat{k})$
$= 10 \times (-1) + 7 \times 0 + (-1) \times 5 = -15$

OR

We have, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=-\hat{i}+\hat{j}$

Now, $(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 0\\ 1 & 0 & 0\end{vmatrix}$

$\Rightarrow\ \lambda\vec{a}+\mu\vec{b}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(0-1)$

$\Rightarrow\ (\lambda+\mu)\hat{i}+(\lambda+\mu)\hat{j}+(\lambda)\hat{k}=-\hat{k}$

On comparing, we get $\lambda = -1$ and $\lambda + \mu = 0$

21. Let the random variable X be defined as the number of spades in a draw of 2 cards successively without replacement, then X can take values 0, 1, 2.

$P(X = 0) = P$(drawing no spade cards)

$=\dfrac{{}^{39}C_2}{{}^{52}C_2}=\dfrac{19}{34}$

$P(X = 1) = P$(drawing one spade and one non-spade card)

$=\dfrac{{}^{13}C_1\times{}^{39}C_1}{{}^{52}C_2}=\dfrac{13}{34}$

$P(X = 2) = P$(drawing both spade cards)

$=\dfrac{{}^{13}C_2}{{}^{52}C_2}=\dfrac{1}{17}$

∴ The probability distribution of number of spades is

X	0	1	2
$P(X)$	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{1}{17}$

22. Here $f(x) = x^3 - 3x^2 + 4x$

$\Rightarrow f'(x) = 3x^2 - 6x + 4 = 3(x^2 - 2x) + 4$
$= 3(x^2 - 2x + 1) - 3 + 4$
$= 3(x - 1)^2 + 1 > 0\ \forall x \in R$

$\Rightarrow f$ is increasing on R.

23. We have, $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$$

So, A is a non-singular matrix and therefore it is invertible.

$$\therefore \quad \text{adj } A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

OR

Given, $2A - 3B + 5C = O$

$$\Rightarrow \; 2A = 3B - 5C \;\Rightarrow\; A = \frac{1}{2}[3B - 5C] \qquad \text{...(i)}$$

$$\text{Now, } 3B - 5C = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\text{From (i), we get } A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}.$$

24. We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x on both sides, we get

$$2\sin y \cos y \frac{dy}{dx} + (-\sin xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{\left(1, \frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2} - 1)}$$

25. The equation of line through $A(-1, 1, -8)$ and $B(5, -2, 10)$ is $\frac{x+1}{5+1} = \frac{y-1}{-2-1} = \frac{z+8}{10+8}$

$$i.e., \; \frac{x+1}{6} = \frac{y-1}{-3} = \frac{z+8}{18} = k(\text{say}) \qquad \text{...(i)}$$

Any point on (i) is given by $(6k - 1, -3k + 1, 18k - 8)$. We know that the coordinates of any point on the ZX-plane is given by $(x, 0, z)$.

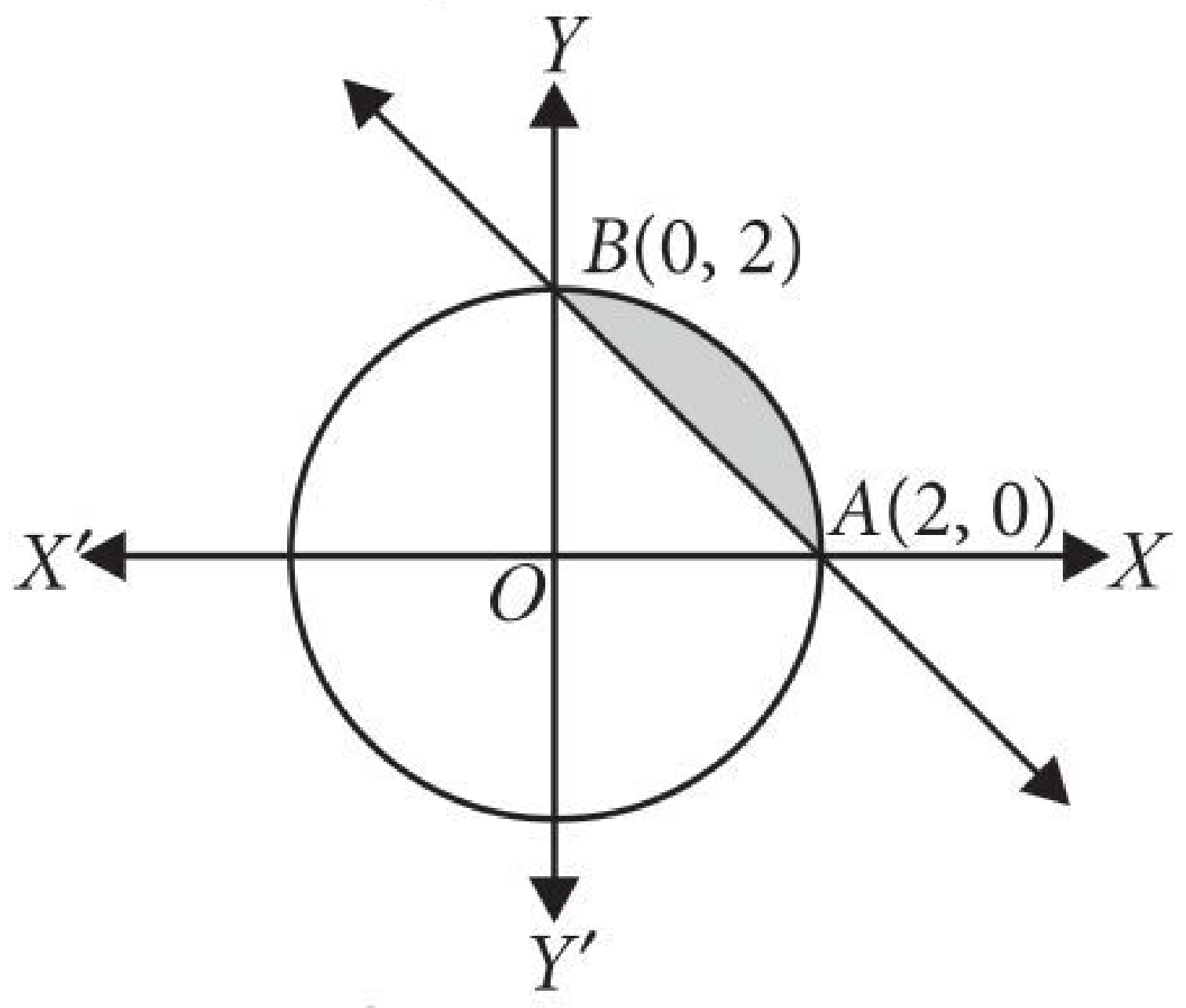
$$\therefore \quad -3k + 1 = 0 \Rightarrow k = \frac{1}{3}$$

Thus the coordinates of the point where the line joining A and B crosses the ZX-plane are

$$\left(6 \times \frac{1}{3} - 1, 0, 18 \times \frac{1}{3} - 8\right) = (1, 0, -2)$$

26. The given curves are

$x^2 + y^2 = 4$ ...(i) and $x + y = 2$ ...(ii)



$$\therefore \quad \text{Required area} = \int_0^2 \left[\sqrt{4 - x^2} - (2 - x)\right]dx$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + 2\sin^{-1}(1) - 4 + 2 - 0 = 2 \cdot \frac{\pi}{2} - 2 = (\pi - 2) \text{ sq.units.}$$

27. We have, $\frac{dy}{dx} + y = \cos x - \sin x$, which is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$, $Q = \cos x - \sin x$

$\therefore \quad$ I.F. $= e^{\int dx} = e^x$

The solution of given differential equation is

$$ye^x = \int e^x(\cos x - \sin x)dx + C$$

$$\Rightarrow \quad ye^x = e^x \cos x + C \Rightarrow y = \cos x + Ce^{-x}$$

28. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}dx$

Put $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx = dt$

$$\therefore \quad I = \frac{2}{3}\int \frac{dt}{\sqrt{a^3 - t^2}} = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$$

$$= \frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right] + C = \frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right] + C$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$$

OR

$$\text{Let } I = \int_0^4 (|x| + |x - 2| + |x - 4|)dx$$

$$= \int_0^2 [x - (x - 2) - (x - 4)]dx + \int_2^4 [x + (x - 2) - (x - 4)]dx$$

$$=\int_0^2(-x+6)dx+\int_2^4(x+2)dx$$

$$=\left[-\frac{x^2}{2}+6x\right]_0^2+\left[\frac{x^2}{2}+2x\right]_2^4$$

$= (-2 + 12 - 0) + [8 + 8 - (2 + 4)]$

$= 10 + 10 = 20$

29. Given $A = \{1, 2, 3, 4,...,9\} \subset N$, the set of natural numbers.

To show : R is an equivalence relation.

(i) Reflexivity : Let (a, b) be an arbitrary element of $A \times A$. Then, we have $(a, b) \in A \times A \Rightarrow a, b \in A$

$\Rightarrow \quad a + b = b + a$

(by commutativity of addition on $A \subset N$)

$\Rightarrow \quad (a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in A \times A$

So, R is reflexive.

(ii) Symmetry: Let $(a, b), (c, d) \in A \times A$ such that $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow b + c = a + d$

$\Rightarrow \quad c + b = d + a$

(by commutativity of addition on $A \subset N$)

$\Rightarrow \quad (c, d)\ R\ (a, b)$.

Thus, $(a, b)\ R\ (c, d)$

$\Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in A \times A$.

So, R is symmetric.

(iii) Transitivity: Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

Now, $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ ...(i)

and $(c, d)\ R\ (e, f) \Rightarrow c + f = d + e$ ...(ii)

Adding (i) and (ii), we get

$(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow \quad a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad (a, b)\ R\ (e, f)$.

So, R is transitive.

$\therefore \quad R$ is an equivalence relation.

Equivalence class for $[(2, 5)]$ is $\{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

30. $\because f(x)$ is continuous at $x = 3$

$\therefore \quad \lim_{x\to3^-} f(x)=\lim_{x\to3^+} f(x)=f(3)$ ...(1)

$\lim_{x\to3^-} f(x)=\lim_{h\to0}(a(3-h)+1)=3a+1$

$\lim_{x\to3^+} f(x)=\lim_{h\to0}(b(3+h)+3)=3b+3$

Also, $f(3) = 3a + 1$

From (1), $3a + 1=3b + 3 \Rightarrow a-b=\frac{2}{3}$,

which is the required relation between a and b.

31. At $x = 1$:

$$f'(1^-)=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{x-1}{x-1}=1$$

$$f'(1^+)=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to1}\frac{2-x-1}{x-1}=\lim_{x\to1}\frac{1-x}{x-1}=-1$$

Since, $f'(1^-) \neq f'(1^+)$

$\therefore \quad f(x)$ is not differentiable at $x = 1$.

At $x = 2$:

$$f'(2^-)=\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to2}\frac{2-x-0}{x-2}=-1$$

$$f'(2^+)=\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to2}\frac{-2+3x-x^2-0}{x-2}=\lim_{x\to2}\frac{(1-x)(x-2)}{x-2}=-1$$

Since, $f'(2^-) = f'(2^+)$

$\therefore \quad f(x)$ is differentiable at $x = 2$.

OR

Refer to Answer 89, page no. 107 of MTG CBSE Champion Mathematics, Class 12.

32. The given curve is

$y = (x^2 - 1)(x - 2)$ ...(i)

It meet x – axis, where $y = 0$

$\therefore \quad (x^2 - 1)(x - 2) = 0 \Rightarrow x = 1, -1, 2$

$\therefore \quad$ We want to find tangents at $(1, 0), (-1, 0)$ and $(2, 0)$

Now from equation (i), we get

$y = x^3 - 2x^2 - x + 2$

$$\Rightarrow \frac{dy}{dx}=3x^2-4x-1$$

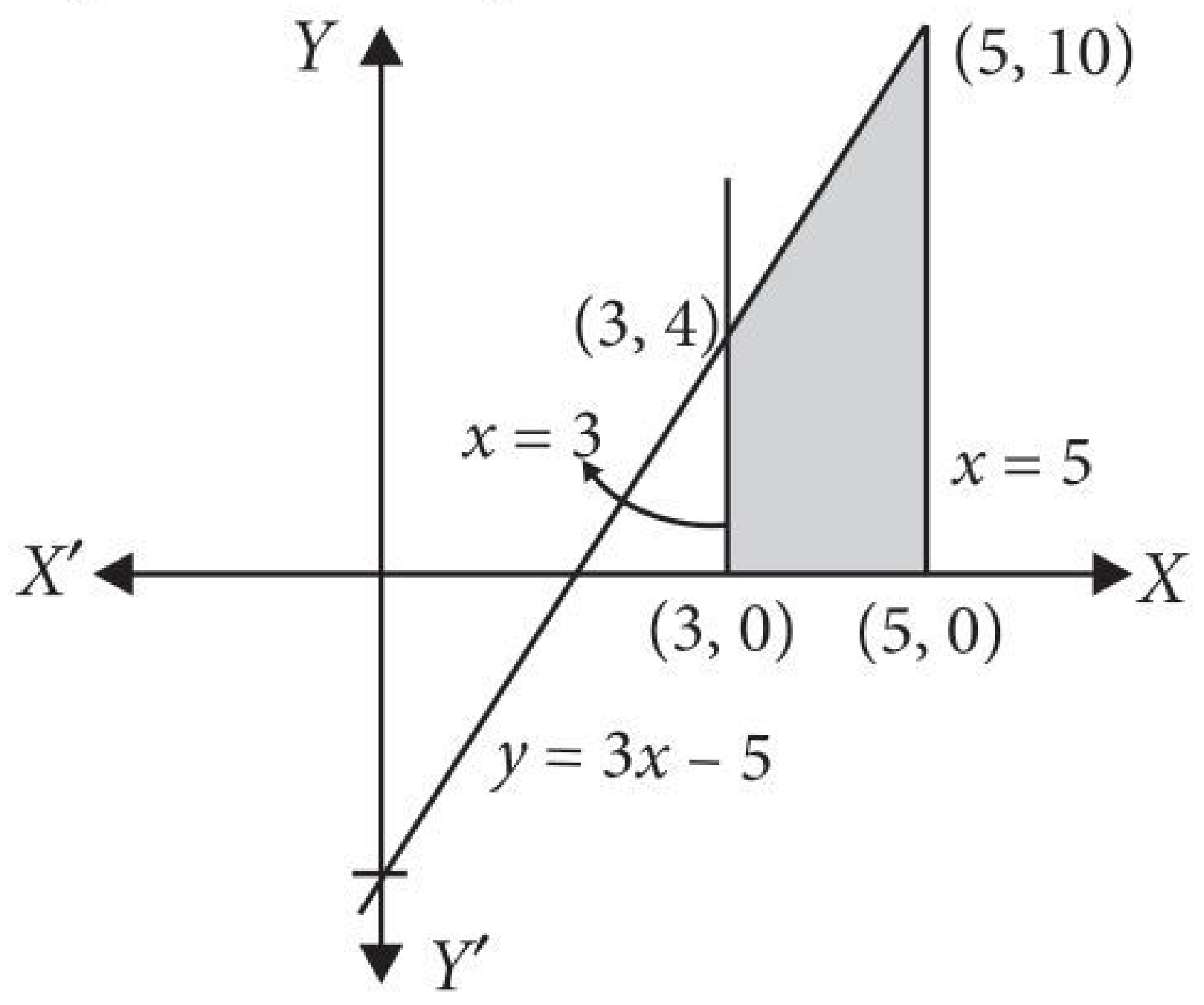
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1}=-2;\left(\frac{dy}{dx}\right)_{x=-1}=6;\left(\frac{dy}{dx}\right)_{x=2}=3$$

Now, tangent at $(1, 0) : y - 0 = -2(x - 1) \Rightarrow 2x + y = 2$

tangent at $(-1, 0) : y - 0 = 6(x + 1) \Rightarrow y = 6x + 6$

tangent at $(2, 0) : y - 0 = 3(x - 2) \Rightarrow y = 3x - 6$

33. We have, $y = 3x - 5$, $y = 0$, $x = 3$, $x = 5$



Required area $= \int_3^5 (3x-5)\,dx = \left[\frac{3x^2}{2} - 5x\right]_3^5$

$= \frac{3}{2}(5^2 - 3^2) - 5(5-3)$

$= \frac{3}{2}(25-9) - 5 \times 2 = 24 - 10 = 14$ sq. units

34. We have, $I = \int \frac{1 + x\cos x}{x(1 - x^2 e^{2\sin x})}dx$

Put $xe^{\sin x} = t \Rightarrow e^{\sin x}(1 + x\cos x)dx = dt$

$\therefore \quad I = \int \frac{dt}{t(1-t^2)} = \int \frac{dt}{t(1+t)(1-t)}$

Let $\frac{1}{t(1+t)(1-t)} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{1-t}$

$\Rightarrow \quad 1 = A(1 - t^2) + B(t - t^2) + C(t + t^2)$

Equating the coefficients of t^2, t and constant terms, we get $A + B - C = 0$, $B + C = 0$ and $A = 1$

Solving these equations, we get

$A = 1, B = \frac{-1}{2}$ and $C = \frac{1}{2}$

$\therefore \quad I = \int \frac{dt}{t(1-t^2)} = \int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{1+t} + \frac{1}{2}\int \frac{dt}{1-t}$

$= \log t - \frac{1}{2}\log|1+t| - \frac{1}{2}\log|1-t| + c$

$= \log|xe^{\sin x}| - \frac{1}{2}\log|1 - x^2 e^{2\sin x}| + c$

OR

Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$= \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2 \cdot 2\sin x\cos x}} = \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{\frac{7}{2}} x \cdot \sin^{\frac{1}{2}} x}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{\frac{7}{2}} x \cdot \tan^{\frac{1}{2}} x \cdot \cos^{\frac{1}{2}} x}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x}}dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

Also $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{4} \Rightarrow t = 1$

$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{(1+t^2)dt}{\sqrt{t}} = \frac{1}{2}\int_0^1 (t^{-\frac{1}{2}} + t^{\frac{3}{2}})\,dt$

$= \frac{1}{2}\left[\frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2}\right]_0^1 = \frac{6}{5}$

35. Given, $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

$\Rightarrow \quad 2xydy = (x^2 + y^2 + 1)dx \Rightarrow 2xydy - y^2dx = (x^2 + 1)dx$

$\Rightarrow \quad xd(y^2) - y^2dx = (x^2 + 1)dx$

$\Rightarrow \quad \frac{xd(y^2) - y^2dx}{x^2} = \left(1 + \frac{1}{x^2}\right)dx \Rightarrow d\left(\frac{y^2}{x}\right) = d\left(x - \frac{1}{x}\right)$

Integrating both sides, we get

$\frac{y^2}{x} = x - \frac{1}{x} + C \Rightarrow y^2 = x^2 - 1 + Cx$

Now, given that $y(1) = 1$

$\therefore \quad 1 = 1 - 1 + C \Rightarrow C = 1$

Thus, curve becomes $y^2 = x^2 - 1 + x$

36. We have maximize

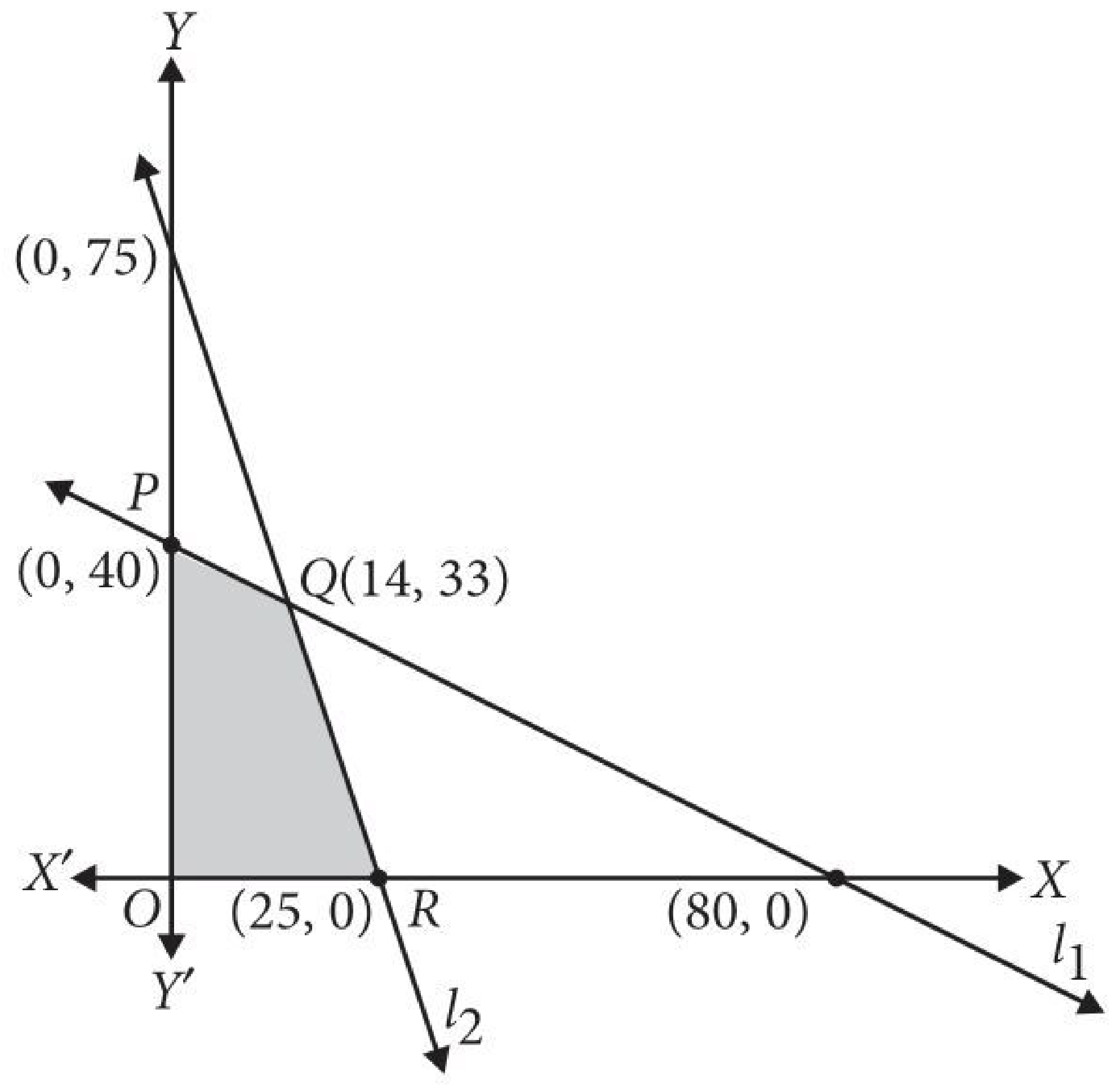
$Z = 4x + 6y$.

Subject to constraints :

$x + 2y \le 80$, $3x + y \le 75$ and $x \ge 0$, $y \ge 0$

Now we draw the graphs of the lines

$l_1 : x + 2y = 80$, $l_2 : 3x + y = 75$ and $x = 0$, $y = 0$.



We obtain shaded region as the feasible region.

The lines l_1 and l_2 intersect at $Q(14, 33)$.

Thus, the vertices of the feasible region are $P(0, 40)$, $Q(14, 33)$, $R(25, 0)$ and $O(0, 0)$.

Corner Points	**Value of $Z = 4x + 6y$**
$P(0, 40)$	240
$Q(14, 33)$	254 (Maximum)
$R(25, 0)$	100
$O(0, 0)$	0

Thus, Z has maximum value 254 at $Q(14, 33)$.

OR

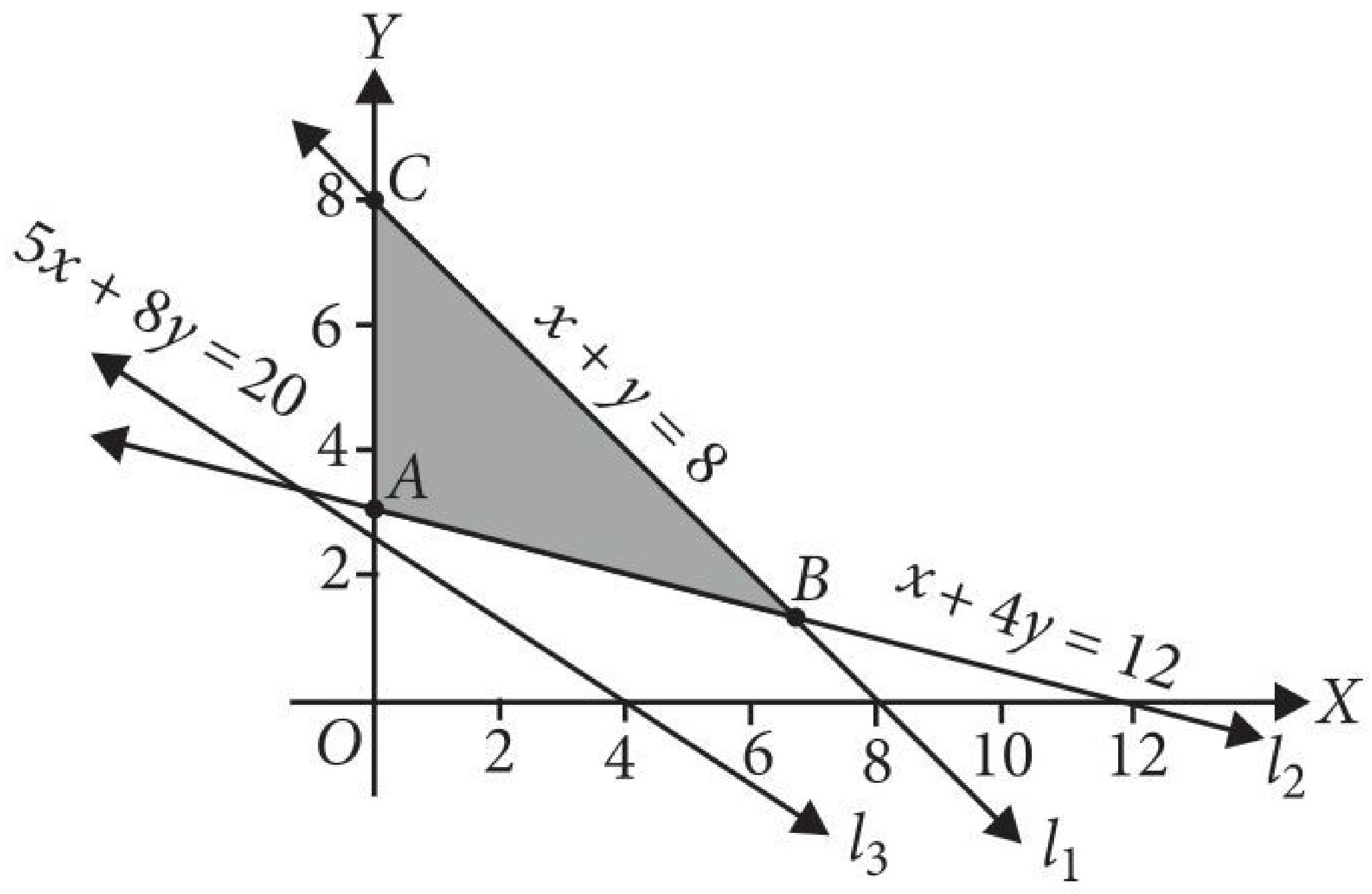
We have minimize $Z = 30x + 20y$.

Subject to constraints :

$$x + y \le 8,\ x + 4y \ge 12,\ 5x + 8y \ge 20,\ x, y \ge 0$$

Now, we draw the graphs of

$l_1 : x + y = 8$, $l_2 : x + 4y = 12$, $l_3 : 5x + 8y = 20$ and $x = 0$, $y = 0$



Shaded region ABC is the required feasible region. $B\left(\frac{20}{3}, \frac{4}{3}\right)$ is the point of intersection of the lines l_1 and l_2.

Thus, the vertices of the feasible region are

$A(0, 3)$, $B\left(\frac{20}{3}, \frac{4}{3}\right)$ and $C(0, 8)$.

Corner Points	**Value of $Z = 30x + 20y$**
$A(0, 3)$	60 (Minimum)
$B(20/3, 4/3)$	226.6
$C(0, 8)$	160

$\therefore$ Z has minimum value 60 at $A(0, 3)$.

37. We have,

$$x + y - z = 3$$
$$2x + 3y + z = 10$$
$$3x - y - 7z = 1$$

The system of equations can be written as $AX = B$

where, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{vmatrix}$

$= 1(-21 + 1) - 1(-14 - 3) - 1(-2 - 9)$

$= -20 + 17 + 11 = 8 \neq 0$

$\therefore$ A^{-1} exists. So, system of equation has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = -20$, $A_{12} = 17$, $A_{13} = -11$, $A_{21} = 8$, $A_{22} = -4$, $A_{23} = 4$, $A_{31} = 4$, $A_{32} = -3$, $A_{33} = 1$

$$\therefore\ \text{adj}A = \begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}\, A = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x = 3, y = 1, z = 1$$

OR

Refer to Answer 127, page no. 81 of MTG CBSE Champion Mathematics, Class 12.

38. The given plane is $2x + 4y - z = 2$ ...(i)

The d.r.'s. of the normal to (i) are 2, 4, -1.

$\therefore$ Eq. of line $\perp$ to (i) through $P(7, 14, 5)$ is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = k\,(\text{say}) \qquad \text{...(ii)}$$

Any point on (ii) is $Q(2k + 7, 4k + 14, -k + 5)$

$\because$ It lies on the plane (i)

$\therefore$ $2(2k + 7) + 4(4k + 14) - (-k + 5) = 2$

$\Rightarrow$ $21k + 65 = 2 \Rightarrow 21k = -63 \Rightarrow k = -3$.

$\therefore$ $Q \equiv (2 \times (-3) + 7, 4 \times (-3) + 14, 3 + 5) \equiv (1, 2, 8)$

This is the foot of $\perp$ on the plane (i)

$$\therefore\ PQ = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2}$$

$$= \sqrt{36 + 144 + 9} = 3\sqrt{21} \text{ units}$$

which is the length of the $\perp$ from P on plane.

Again let $R(x_1, y_1, z_1)$ be the image of P in the plane (i). Then Q is the mid point of PR.

$$\therefore\ \frac{x_1 + 7}{2} = 1,\ \frac{y_1 + 14}{2} = 2,\ \frac{z_1 + 5}{2} = 8$$

$\Rightarrow$ $x_1 = -5$, $y_1 = -10$, $z_1 = 11$

$\Rightarrow$ $R \equiv (-5, -10, 11)$

OR

Refer to Answer 60, page no. 269 of MTG CBSE Champion Mathematics, Class 12.

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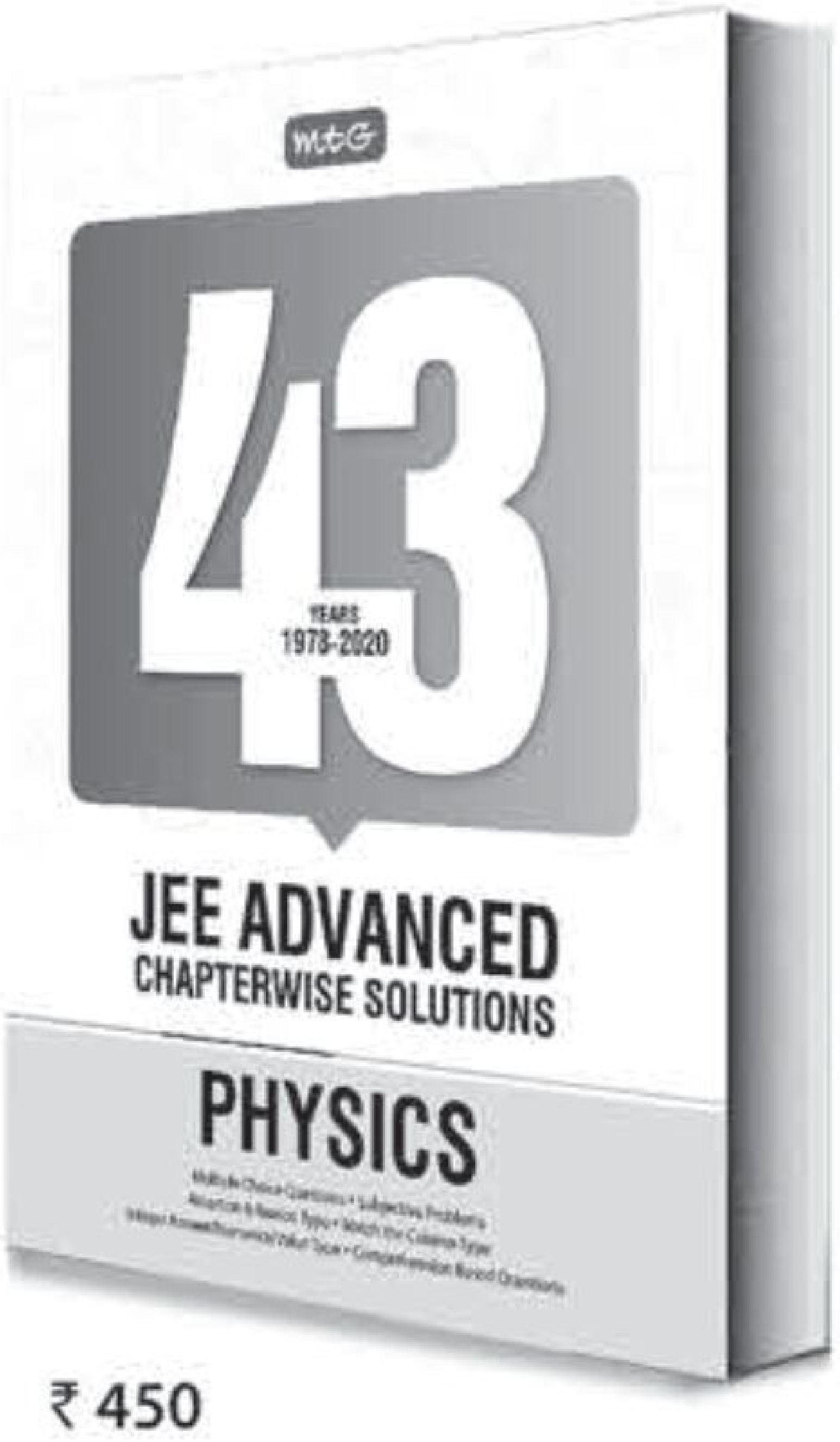
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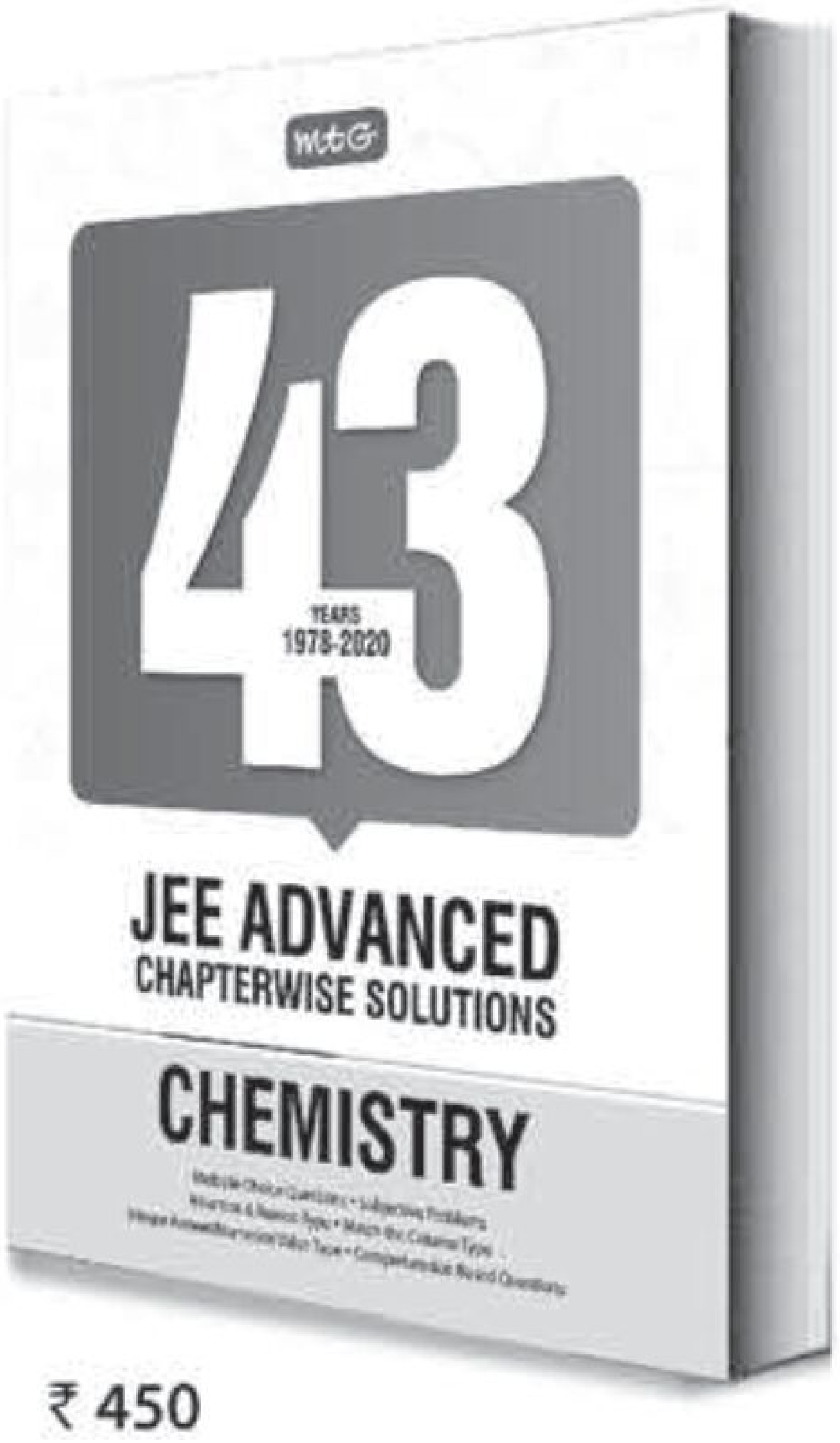
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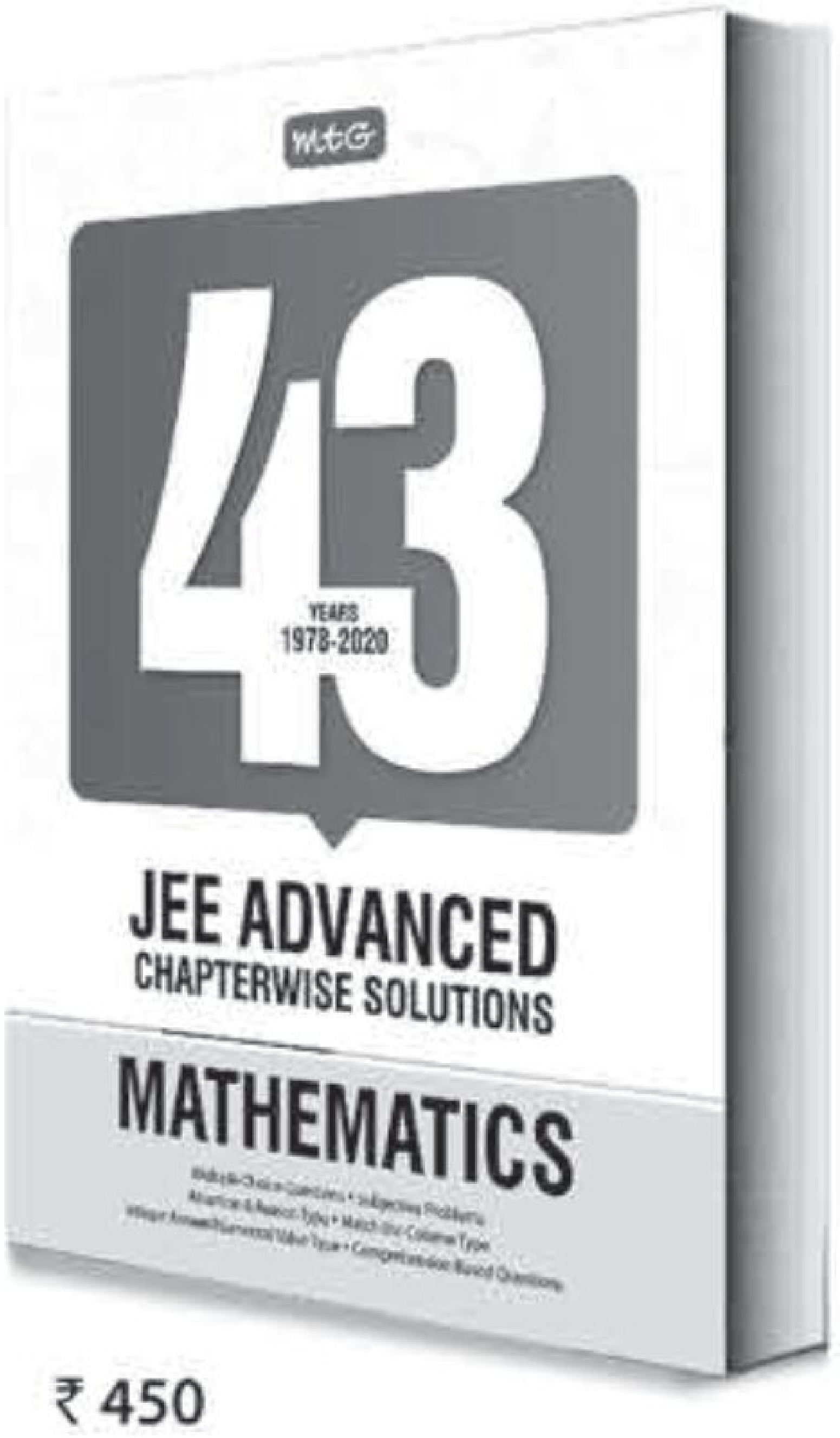
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MONTHLY TEST DRIVE

XII

This specially designed column enables students to self analyse their extent of understanding of all chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The largest interval lying in $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ for which the function, $f(x)=4^{-x^2}+\cos^{-1}\left(\frac{x}{2}-1\right)+\log(\cos x)$ is defined, is

(a) $\left[-\frac{\pi}{4},\frac{\pi}{2}\right)$ (b) $\left[0,\frac{\pi}{2}\right)$

(c) $[0, \pi]$ (d) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

2. The value of $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) none of these

3. If from each of 3 boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn then the probability of drawing 2 white and 1 black ball is

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$ (c) $\frac{1}{32}$ (d) $\frac{3}{16}$

4. $\sin^{-1}(\cos\sin^{-1}x)+\cos^{-1}(\sin\cos^{-1}x)=$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

5. Let $f:[0, 2]\to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x)=\int_0^{x^2} f(\sqrt{t})dt$, for $x\in[0, 2]$. If $F'(x)=f'(x)$, for all $x\in(0, 2)$, then $F(2)$ is

(a) e^2-1 (b) e^4-1 (c) $e-1$ (d) e^4

6. Let $f(x)$ be a function defined on $[0, 1]$ such that

$$f(x)=\begin{cases} x, & \text{if } x\in Q \\ 1-x, & \text{if } x\notin Q. \end{cases}$$

Then for all $x\in[0, 1]$, $(fof)(x)$ is

(a) constant (b) $1+x$

(c) x (d) x^2

One or More Than One Option(s) Correct Type

7. If $\int\frac{(2x-1)dx}{x^4-2x^3+x+1}=A\tan^{-1}\left[\frac{f(x)}{\sqrt{3}}\right]+C$, then

(a) $A=\frac{2}{\sqrt{3}}$ (b) $A = 2$

(c) $f(x) = 2x^2-2x-1$ (d) $f(x) = 2x^2+2x+1$

8. Non-zero vectors $\vec{a}$ and $\vec{b}$ are equally inclined to vector $\vec{c}$, if $\vec{c}$ is equal to

(a) $\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b}$

(b) $\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{b}$

(c) $\frac{|\vec{a}|}{|\vec{a}|+2|\vec{b}|}\vec{a}+\frac{|\vec{b}|}{|\vec{a}|+2|\vec{b}|}\vec{b}$

(d) $\frac{|\vec{b}|}{2|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{a}|}{2|\vec{a}|+|\vec{b}|}\vec{b}$

9. Let $f(x)=\begin{cases}\int_0^x (1+|1-t|)dt, & x>2 \\ 5x+1, & x\le 2\end{cases}$, then

(a) $f(x)$ is not continuous at $x = 2$

(b) $f(x)$ is continuous but not differentiable at $x = 2$

(c) $f(x)$ is differentiable everywhere

(d) the right derivative of $f(x)$ at $x = 2$ does not exist.

10. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0, a > 0, 0 \le x \le 2a$, then

(a) $\phi(x)$ increases in $(a, 2a)$

(b) $\phi(x)$ increases in $(0, a)$

(c) $\phi(x)$ decreases in $(a, 2a)$

(d) $\phi(x)$ decreases in $(0, a)$

11. Which of the following functions are periodic?

(a) $f(x) = \sin x + |\sin x|$

(b) $g(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$

(c) $h(x) = \max(\sin x, \cos x)$

(d) $p(x) = [x] + \left[x + \dfrac{1}{3}\right] + \left[x + \dfrac{2}{3}\right] - 3x + 10$, where [.] denotes the greatest integer function.

12. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \ne 0$) satisfies the equation $A^2 + kI = O$, then

(a) $a + d = 0$ (b) $k = -|A|$

(c) $k = |A|$ (d) none of these

13. If $f(x) = \begin{cases} \dfrac{\tan^2\{x\}}{(x^2 - [x]^2)} & , \text{ for } x > 0 \\ 1 & , \text{ for } x = 0 \\ \sqrt{\{x\} \cot\{x\}} & , \text{ for } x < 0 \end{cases}$

where, $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then which of the following is not correct?

(a) $\lim\limits_{x \to 0} f(x) = 1$ (b) $\lim\limits_{x \to 0^-} f(x) = \cot 1$

(c) $\cot^{-1}\left(\lim\limits_{x \to 0^-} f(x)\right)^2 = 1$ (d) f is continuous at $x = 0$

Comprehension Type

Let two planes $P_1 : 2x - y + z = 2$ and $P_2 : x + 2y - z = 3$ are given.

14. The equation of the acute angle bisector of planes P_1 and P_2 is

(a) $x - 3y + 2z + 1 = 0$ (b) $3x + y - 5 = 0$

(c) $x + 3y - 2z + 1 = 0$ (d) $3x + z + 7 = 0$

15. The image of plane P_1 in the plane mirror P_2 is

(a) $x + 7y - 4x + 5 = 0$ (b) $3x + 4y - 5z + 9 = 0$

(c) $7x - y + 2z - 9 = 0$ (d) None of the above

Matrix Match Type

16. Match the differential equations with their solutions.

Column-I		Column-II	
P.	$(1 + x\sqrt{x^2 + y^2})dx + (-1 + \sqrt{x^2 + y^2})y\,dy = 0$	1.	$xe^{\sin\frac{y}{x}} = C$
Q.	$(x - 1)\,dy + y\,dx = x(x - 1)y^{1/3}\,dx$	2.	$x - \dfrac{y^2}{2} + \dfrac{1}{3}(x^2 + y^2)^{3/2} = C$
R.	$[y(1 + x^{-1}) + \sin y]\,dx + (x + \log x + x\cos y)\,dy = 0$	3.	$y^{2/3}(x - 1)^{2/3} = \dfrac{(x-1)^{8/3}}{4} + \left(\dfrac{2}{5}\right)(x - 1)^{5/3} + C$
S.	$x\cos\dfrac{y}{x}dy = \left(y\cos\dfrac{y}{x} - x\right)dx$	4.	$xy + y\log x + x\sin y = C$

	P	Q	R	S
(a)	3	1	2	4
(b)	1	3	4	2
(c)	1	4	2	3
(d)	2	3	4	1

Numerical Value Type

17. If $f : R \to R$, $g : R \to R$ and $h : R \to R$ is such that $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$, then the value of $[ho(gof)]\,(x)$, if $x = \dfrac{\sqrt{\pi}}{2}$, will be_____.

18. The vectors $x\hat{i} - 3\hat{j} + 7\hat{k}$ and $\hat{i} + y\hat{j} - z\hat{k}$ are collinear, then the value of $\dfrac{-7xy^2}{z}$ is equal to_____.

19. The distance between the skew lines $\dfrac{x}{4} = \dfrac{y-7}{-3} = \dfrac{z-7}{-2}$ and $\dfrac{x-4}{2} = \dfrac{y+6}{3} = \dfrac{z-34}{-10}$ is_____.

20. The area bounded by the y-axis, the tangent and normal to the curve $x + y = x^y$ at the point where the curve cuts the x-axis is _____.

◆◆

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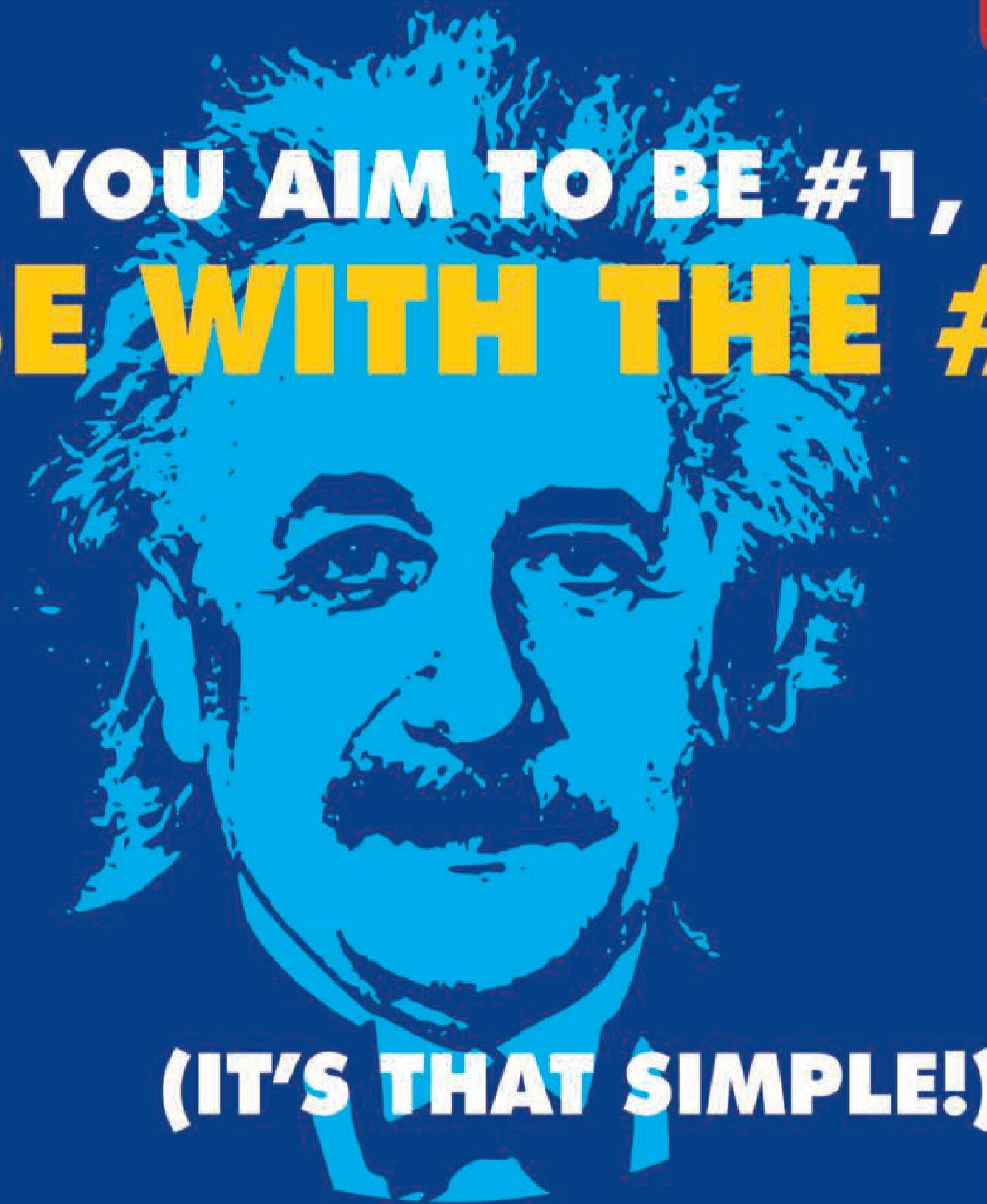
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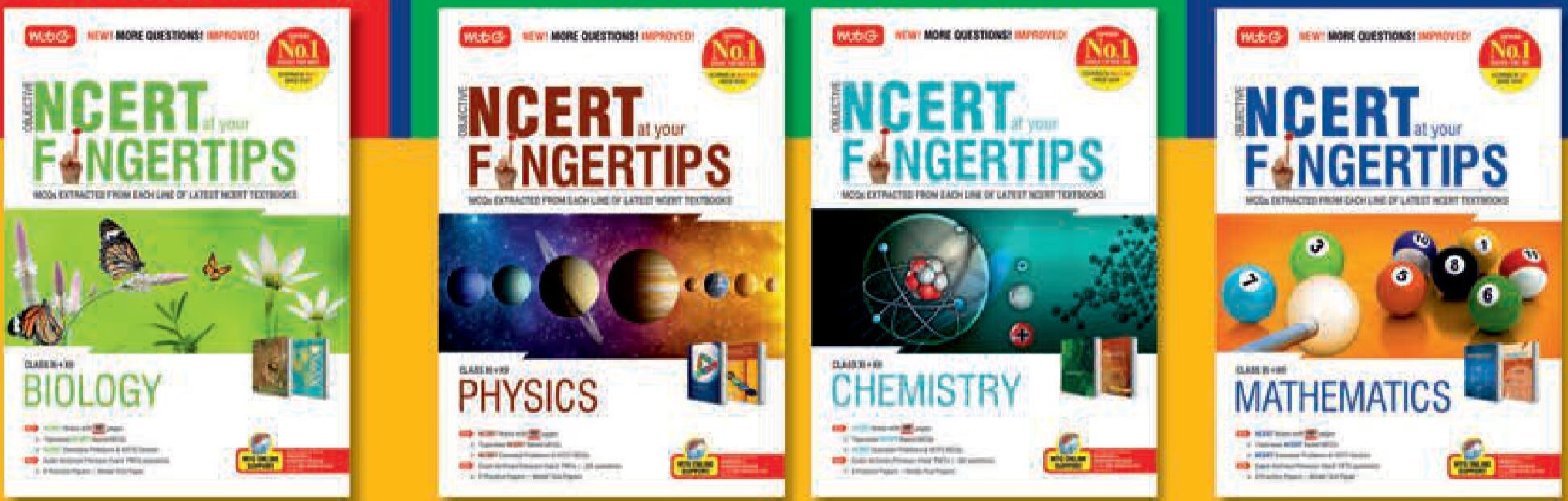
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